

Decompositions of Analytic 1-Manifolds

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Abstract

In [1], connected analytic 1-dimensional submanifolds with boundary have been classified w.r.t. their symmetry under a given regular Lie group action on an analytic manifold. It was shown that each such submanifold is either free or analytically diffeomorphic to the unit circle or some interval via the exponential map. In this paper, we show that each free connected analytic 1-submanifold naturally splits into symmetry free segments, mutually and uniquely related by the group action. This is proven under the assumption that the action is non-contractive, which is even less restrictive than regularity.

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1 Introduction

In [1] it was shown that, given a regular Lie group action $\varphi: G \times M \rightarrow M$ on an analytic manifold M , an analytic curve in M or a connected analytic 1-submanifold with boundary of M , is either exponential or free. Here,

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- An analytic curve $\gamma: D \rightarrow M$ is said to be exponential iff

$$\gamma: t \mapsto \exp(\rho(t) \cdot \vec{g}) \cdot x \quad \forall t \in D$$

holds for some $x \in M$, some $\vec{g} \in \mathfrak{g}$, and some analytic map $\rho: D \rightarrow D' \subseteq \mathbb{R}$.

Here, in the non-constant analytic case, $\vec{g} \in \mathfrak{g} \setminus \mathfrak{g}_x$ is uniquely determined up to scaling by some $\lambda \neq 0$, and addition of an element in \mathfrak{g}_γ ; the Lie algebra of the stabilizer $G_\gamma = \bigcap_{t \in D} G_{\gamma(t)}$ of γ .

- A connected analytic 1-submanifold¹ (S, ι) is said to be exponential iff it is either analytically diffeomorphic to $U(1)$ or to some interval via

$$e^{i\phi} \mapsto \iota^{-1}(\exp(\phi \cdot \vec{g}) \cdot x) \quad \text{or} \quad t \mapsto \iota^{-1}(\exp(t \cdot \vec{g}) \cdot x),$$

respectively, for some $\vec{g} \in \mathfrak{g} \setminus \mathfrak{g}_x$.

Here, (for fixed interval in the second case) $\vec{g} \in \mathfrak{g} \setminus \mathfrak{g}_x$ is uniquely determined up to addition of an element in \mathfrak{g}_S ; the Lie algebra of the stabilizer $G_S = \bigcap_{z \in S} G_{\iota(z)}$ of S .

In addition to that, it was shown that each free immersive analytic curve naturally decomposes into symmetry free subcurves, mutually and uniquely related by the symmetry action. In this complete paper, we prove an analogous result for connected analytic 1-submanifolds with boundary, i.e., we show that each such submanifold decomposes naturally into symmetry free segments, mutually and uniquely related by the group action. More precisely, let us say that φ is non-contractive iff

- The map $\varphi_g: M \rightarrow M, x \mapsto \varphi(g, x)$ is analytic for each $g \in G$.
- No subset $C \subseteq M$ with $|C| \geq 2$ can be contracted to some $x \in M - C$, i.e., we find a neighbourhood U of x with $g \cdot C \not\subseteq U$ for each $g \in G$.

The second point is equivalent to satedness from Definition 2.14 in [1], so that Remark 2.15 in [1] shows that φ is non-contractive if each φ_g is analytic, and if, e.g.,

- There exists some G -invariant continuous metric on M .
- The maps $\varphi_x: g \mapsto \varphi(g, x)$ are proper for each $x \in M$.
- M is a topological group with

$$\varphi(g, x) = \phi(g) \cdot x \quad \forall g \in G, \quad \forall x \in M,$$

for some continuous group homomorphism $\phi: G \rightarrow M$.

Now, a connected analytic 1-submanifold with boundary (S, ι) of M is said to be free iff it admits a free segment, i.e., a connected subset $\Sigma \subseteq S$ with non-empty interior, such that

$$g \cdot \iota|_\Sigma \sim_\circ \iota|_\Sigma \quad \text{for } g \in G \quad \implies \quad g \cdot \iota|_\Sigma = \iota|_\Sigma \quad \iff \quad g \in G_S,$$

for $G_S := \bigcap_{z \in S} G_{\iota(z)}$ the stabilizer of $\iota(S)$.

Here, we write $g \cdot \iota|_\Sigma \sim_\circ \iota|_{\Sigma'}$ for segments $\Sigma, \Sigma' \subseteq S$ iff $g \cdot \iota(\mathcal{O}) = \iota(\mathcal{O}')$ holds for open connected subsets $\mathcal{O} \subseteq \Sigma$ and $\mathcal{O}' \subseteq \Sigma'$ which are contained in the interior $\text{int}[S]$ of S , and on which ι is an embedding. We say that a free segment Σ is maximal iff $\Sigma \subseteq \Sigma'$ for a free segment $\Sigma' \subseteq S$ implies that $\Sigma = \Sigma'$ holds; and obtain, cf. Theorem 3.17

Theorem

Let φ be non-contractive, and (S, ι) free but not a free segment by itself. Then, S either admits a unique z -decomposition or a compact maximal segment properly contained in $\text{int}[S]$. Here, the first case cannot occur if S is compact without boundary; and in the second case, S is either positive or negative, and admits a unique Σ -decomposition for each (necessarily compact) maximal segment $\Sigma \subset \text{int}[S]$.

¹This means that S is a connected analytic 1-dimensional manifold with boundary, and that $\iota: S \rightarrow M$ is an injective analytic immersion.

Now, in order to understand the above theorem and its uniqueness statements in more detail, let us first observe that S is either homeomorphic to $U(1)$ or to some interval, hence compact without boundary iff $S \cong U(1)$ holds. Then,

If S is homeomorphic to an interval, by a z -decomposition of S , we understand a class $[g] \in G/G_S - \{[e]\}$ with $g \in G_{\iota(z)}$, such that $g \cdot \iota(\mathcal{K}_-) = \iota(\mathcal{K}_+)$ holds for compact segments $\mathcal{K}_\pm \subseteq S$ with $\mathcal{K}_- \cap \mathcal{K}_+ = \{z\}$. Then, the unique boundary segments Σ_\pm of S , for which²

$$S = \Sigma_- \cup \Sigma_+ \quad \Sigma_- \cap \Sigma_+ = \{z\} \quad \text{and} \quad \mathcal{K}_\pm \subseteq \Sigma_\pm$$

holds, are the only maximal ones, cf. Lemma 3.14, and exactly one of the following two situations holds:

$$\begin{aligned} \iota^{-1} \circ (g \cdot \iota|_{C_-}) : \quad \Sigma_- \supset C_- &\rightarrow \Sigma_+ & \text{is an analytic diffeomorphism,} \\ \iota^{-1} \circ (g \cdot \iota|_{\Sigma_-}) : \quad \Sigma_- &\rightarrow C_+ \subseteq \Sigma_+ & \text{is an analytic diffeomorphism,} \end{aligned}$$

whereby $C_\pm \subseteq \Sigma_\pm$ are segments with $\mathcal{K}_\pm \subseteq C_\pm$. In addition to that, we have

$$g' \cdot \iota|_{\Sigma_-} \sim_\circ \iota \quad \text{for } g' \in G \quad \implies \quad [g'] \in \{[e], [g]\}, \quad (1)$$

so that a proper translate of $\iota(\Sigma_-)$ can overlap $\iota(S)$ in exactly one way. The respective uniqueness statement in the above theorem (Theorem 3.17) then has to be understood in that way that the class $[g]$, and the point $z \in S$ are uniquely determined.

Example

Let φ be the canonical action of $G := \text{SO}(2)$ on $M := \mathbb{R}^2$, $S := I$ an open interval containing 0, and $\iota : I \rightarrow \mathbb{R}^2$, $t \mapsto (t, t^3)$. Then, S admits the 0-decomposition $[R(\pi)]$, for $R(\pi)$ the rotation by the angle π . Moreover, $\Sigma_- = I \cap (-\infty, 0]$ and $\Sigma_+ = I \cap [0, \infty)$ holds, whereby we can define $\mathcal{K}_- := [-\epsilon, 0]$ and $\mathcal{K}_+ := [0, \epsilon]$ for each $\epsilon > 0$ with $\pm\epsilon \in I$.

Now, if S admits a compact maximal segment $\Sigma \subset \text{int}[S]$, each translate of $\iota(\Sigma)$ overlaps $\iota(S)$ in a unique way, cf. (2) and (3). More precisely,

- I) If S is compact without boundary, a Σ -decomposition (cf. Definition 3.4) is a collection of compact maximal segments $\Sigma_0, \dots, \Sigma_n$ and classes $[g_0], \dots, [g_n] \in G/G_S$ with

$$\Sigma_0 = \Sigma \quad S = \bigcup_{k=0}^n \Sigma_k \quad \text{and} \quad g_k \cdot \iota(\Sigma_0) = \iota(\Sigma_k) \quad \forall 0 \leq k \leq n,$$

such that

- If $n = 1$, then $\Sigma_0 \cap \Sigma_1 = \{z_-, z_+\}$ consists of the two boundary points of Σ_0 .
- If $n \geq 2$, then $\Sigma_p \cap \Sigma_q$ is singleton for $|p - q| \in \{1, n\}$, and empty otherwise.

As a consequence, the classes $[g_0], \dots, [g_n]$ are mutually different, and we have $[g_0] = [e]$, as well as $[g_1] = [g_1^{-1}]$ for $n = 1$.

In addition to that, in analogy to (1), we have³ (cf. Lemma 3.6)

$$g \cdot \iota(\Sigma_0) = \iota(\Sigma') \quad \implies \quad [g] = [g_k] \quad \text{and} \quad \Sigma' = \Sigma_k \quad \text{for } 0 \leq k \leq n \quad \text{unique}, \quad (2)$$

for each segment $\Sigma' \subseteq S$, and each $g \in G$. In particular, for $n = 1$, there exists no other Σ -decomposition of S ; and for $n \geq 2$, the only other Σ -decomposition is given by

$$\overline{\Sigma}_k := \Sigma_{\zeta(k)} \quad \text{and} \quad [\overline{g}_k] := [g_{\zeta(k)}] \quad \forall 1 \leq k \leq n$$

for $\zeta \in S_n$ defined by $\zeta(k) = n - (k - 1)$ for $k = 1, \dots, n$.

Finally, the number of segments occurring in such a decomposition is the same for each maximal segment contained in S , cf. Lemma 3.7.

²By a boundary segment of a segment Σ , we understand a segment Σ' with $\Sigma' \subset \Sigma$, such that $\Sigma - \Sigma'$ is a segment as well.

³By Lemma 2.5, $g \cdot \iota \sim_\circ \iota$ already implies that $g \cdot \iota(S) = \iota(S)$ holds, whereby $g \cdot \iota$ and ι are embeddings as S is compact.

II) If S is homeomorphic to an interval, by a Σ -decomposition (cf. Definition 3.8), we understand a pair $(\{\Sigma_n\}_{n \in \mathbf{n}}, \{[g_n]\}_{n \in \mathbf{n}})$ for

$$\mathbf{n} = \{n \in \mathbb{Z} \mid \mathbf{n}_- \leq n \leq \mathbf{n}_+\} \quad \text{with} \quad \mathbf{n}_-, \mathbf{n}_+ \in \mathbb{Z}_{\neq 0} \sqcup \{-\infty, \infty\} \quad \text{for} \quad \mathbf{n}_- < 0 < \mathbf{n}_+,$$

consisting of classes $[g_n] \in G/G_S$, and segments Σ_n on which ι is an embedding. Moreover, $\Sigma_0 = \Sigma$ holds, and

- $\Sigma_p \cap \Sigma_q$ is singleton for $|p - q| = 1$, and empty otherwise,
- $g_n \cdot \iota(\Sigma_0) = \iota(\Sigma_n)$ holds for all $\mathbf{n}_- < n < \mathbf{n}_+$, as well as

$$g_{\mathbf{n}_-} \cdot \iota(\Sigma_-) = \iota(\Sigma_{\mathbf{n}_-}) \quad \text{iff} \quad \mathbf{n}_- > -\infty \quad \text{and} \quad g_{\mathbf{n}_+} \cdot \iota(\Sigma_+) = \iota(\Sigma_{\mathbf{n}_+}) \quad \text{iff} \quad \mathbf{n}_+ < \infty.$$

Here, $\Sigma_{\mathbf{n}_\pm}$ are boundary segments of S , and the Σ_\pm either equal, or are boundary segments of, Σ_0 .

Then, each Σ_n is free, as well as compact maximal if $g_n \cdot \iota(\Sigma_0) = \iota(\Sigma_n)$ holds. In addition to that, we have $S = \bigcup_{n \in \mathbf{n}} \Sigma_n$, (cf. Lemma and Remark 3.9.3), $[g_0] = [e]$; and $[g_m] = [g_n]$ for $m \neq n$ implies $m, n \in \{\mathbf{n}_-, \mathbf{n}_+\}$ with $-\infty < \mathbf{n}_- < \mathbf{n}_+ < \infty$.

Finally, for each $g \in G$, we have, cf. Lemma 3.10

$$(g \cdot \iota|_{\Sigma_0})(\mathcal{O}) = \iota(\mathcal{O}') \quad \implies \quad [g] = [g_n] \quad \text{and} \quad \mathcal{O}' \subseteq \Sigma_n \quad \text{for} \quad n \in \mathbf{n} \quad \text{unique} \quad (3)$$

for $\mathcal{O}, \mathcal{O}' \subseteq \text{int}[S]$ open and connected, such that $\iota|_{\mathcal{O}}$ and $\iota|_{\mathcal{O}'}$ are embeddings. In particular, a Σ -decomposition is unique up to a reordering of the form, cf. Lemma 3.10

$$(\{\overline{\Sigma}_n\}_{n \in \overline{\mathbf{n}}}, \{[\overline{g}_n]\}_{n \in \overline{\mathbf{n}}}) \quad \text{with} \quad \overline{\Sigma}_n := \Sigma_{-n} \quad \text{and} \quad [\overline{g}_n] := [g_{-n}] \quad \text{for each} \quad n \in \mathbf{n},$$

whereby $\overline{\mathbf{n}} := \{n \in \mathbb{Z} \mid \overline{\mathbf{n}}_- \leq n \leq \overline{\mathbf{n}}_+\}$ holds for $\overline{\mathbf{n}}_\pm := -\mathbf{n}_\mp$.

Finally, it remains to explain what positivity and negativity means. For this, let $\Sigma \subset \text{int}[S]$ be compact maximal, with what we are in the situation of I) or II), depending on whether S is homeomorphic to $U(1)$ or to some interval, respectively. We denote the two boundary points of Σ by z_\pm , whereby we can assume that $z_+ \in \Sigma_0 \cap \Sigma_1$ holds.⁴ Then, Σ is said to be positive or negative iff $g_{\pm 1} \notin G_{z_\pm}$ or $g_{\pm 1} \in G_{z_\pm}$ holds, respectively, whereby we define $g_{-1} := g_n$ iff we are in the situation of I). Figuratively speaking,

- positivity means that g_\pm shifts $\iota(\Sigma)$ in such a way that $\iota(z_\mp)$ is mapped to $\iota(z_\pm)$,
- negativity means that g_\pm flips $\iota(\Sigma)$ at $\iota(z_\pm)$.

It follows that Σ is either positive or negative, cf. Lemma 3.2, and we even have that each compact maximal segment $\Sigma' \subset \text{int}[S]$ is positive/negative iff one such segment is positive/negative, cf. Lemma 3.16. Then,

- If Σ is positive, we have, cf. Lemma 3.19

$$\begin{aligned} \text{I):} \quad & [g_k] = [g^k] \quad \text{for} \quad k = 0, \dots, n \quad \text{for each} \quad g \in [g_1], \\ \text{II):} \quad & [g_n] = [g^n] \quad \forall n \in \mathbf{n} \quad \text{for each} \quad g \in [g_1]. \end{aligned}$$

Here, up to inversion,⁵ the class $[g_1]$ is the same for each decomposition which corresponds to a compact maximal segment contained in $\text{int}[S]$. In addition to that, S admits a lot more compact maximal segments than just those occurring in the Σ -decomposition of S . For instance, each $z \in \text{int}[S]$ is contained in the interior of some positive segment $\Sigma_z \subset \text{int}[S]$, cf. Corollary 3.21.

For instance,

- I) Let $M = S := U(1)$, and define G to be the discrete subgroup of $U(1)$ generated by $g := e^{i2\pi/n}$, and acting via multiplication from the left. Then, $\Sigma := e^{iK}$ is positive for each $K = [t, t + 2\pi/n]$, and admits the obvious Σ -decomposition of S , for which $[g_1] = [g]$ holds.

⁴This intersection is only non-singleton iff $S \cong U(1)$ and $n = 1$ holds, in which case we have $\Sigma_0 \cap \Sigma_1 = \{z_-, z_+\}$.

⁵This means that each such class either equals $[g_1]$ or $[g_1^{-1}]$; more details can be found in Subsection 3.4.2.

II) Let $G := \mathbb{R}$ act via $\varphi(t, (x, y)) := (t+x, y)$ on $M := \mathbb{R}^2$, and define $S := \mathbb{R}$ as well as $\iota: t \mapsto (t, \sin(t))$. Then, S admits the positive segments $[t, t+2\pi]$ with $t \in \mathbb{R}$, each giving rise to a decomposition of S for which $[g_1] = [2\pi]$ holds.

► If Σ is negative, the segments occurring in the Σ -decomposition of S are the only maximal ones, cf. Lemma 3.16. In addition to that, we have, cf. (27) and (26)

$$\begin{aligned} \text{I):} \quad & [g_k] = [g_{\sigma(1)} \cdot \dots \cdot g_{\sigma(k)}] & \forall 1 \leq k \leq n & \quad \text{for} \quad g_{-1} := g_n, \\ \text{II):} \quad & [g_n] = [g_{\sigma(\text{sign}(n))} \cdot \dots \cdot g_{\sigma(n)}] & \forall n \in \mathbf{n} - \{0\}, \end{aligned}$$

with $\sigma: \mathbb{Z}_{\neq 0} \rightarrow \{-1, 1\}$, defined by

$$\sigma(n) := \begin{cases} (-1)^{n-1} & \text{if } n > 0 \\ (-1)^n & \text{if } n < 0. \end{cases}$$

Thus, we have $[g_2] = [g_1 \cdot g_{-1}]$, $[g_3] = [g_1 \cdot g_{-1} \cdot g_1]$, and so on. In particular, in the situation of I), the integer n must be odd, cf. Corollary 3.18.

For instance,

- I) Let $S := U(1) \subseteq M := \mathbb{R}^2$, and G be the discrete group generated by the reflection at the x_2 -axis. Then, $\Sigma_0 = e^{iK_0}$ and $\Sigma_1 = e^{iK_1}$ are negative for $K_0 = [-\pi/2, \pi/2]$ and $K_1 = [\pi/2, 3\pi/4]$.
Similarly, if G is the discrete group generated by the reflection at the x_1 - and the x_2 -axis, then $\Sigma_k = e^{iK_k}$ is negative for $K_k = [k \cdot \pi/4, (k+1) \cdot \pi/4]$ for $k = 0, \dots, 3$, and the above formula for the classes $[g_k]$ is easily verified.
- II) Let the euclidean group $\mathbb{R}^2 \rtimes \text{SO}(2)$ act on \mathbb{R}^2 in the canonical way, and define $S := \mathbb{R}$ as well as $\iota: \mathbb{R} \ni t \mapsto (t, \sin(t))$. Then, $\Sigma = [0, \pi]$ is negative with $[g_{-1}]$ and $[g_1]$ the classes of the rotations by π around $(0, 0)$ and $(\pi, 0)$, respectively.

This paper is organized as follows:

In Section 2, we fix the conventions and collect the basic facts and definitions that we will need in Section 3. There, we first prove some basic statements concerning compact maximal segments, and then show that each such segment Σ with $\Sigma \subset \text{int}[S]$ admits a Σ -decomposition of S . In the third part of Section 3, we will show that S admits a unique z -decomposition iff no compact maximal $\Sigma \subseteq \text{int}[S]$ exists; and in the last part, we will work out the positive and the negative case.

2 Preliminaries

In this section, we fix the conventions, and provide some basic facts and definitions concerning connected analytic 1-manifolds with boundary, as well as free segments.

2.1 Conventions

A curve is a continuous map $\gamma: D \rightarrow X$ between an interval D , and a topological space X . Here, an interval $D \subseteq \mathbb{R}$ will always be assumed to have non-empty interior $\text{int}[D]$; and if we write I, J or K instead of D , we will always mean that I, J are open, and that K is compact.

Manifolds, and those with boundary, will always be assumed to be Hausdorff second-countable and analytic; whereby domains of charts will always be assumed to be non-empty, open, and connected.

If M is a manifold, an **analytic 1-submanifold** of M is a pair (S, ι) consisting of a connected analytic 1-manifold S with boundary, together with an injective analytic immersion $\iota: S \rightarrow M$. If (U, ψ) is a chart of S , we will use the convention that $\psi(U) \subseteq [0, \infty)$ holds if U contains some boundary point of S . In addition to that, we will require that $\iota(U) \subseteq V$ holds for some chart (V, ϕ) of M , so that

$$\gamma_\psi: \psi(U) \rightarrow M, \quad t \mapsto \iota \circ \psi^{-1}(t)$$

is an **analytic immersive curve**; i.e., $\gamma_\psi = \tilde{\gamma}_\psi|_{\psi(U)}$ holds for an analytic immersive curve (analytic immersive extension) $\tilde{\gamma}_\psi: I \rightarrow M$, defined on an open interval I .

By a **segment**, we will understand a connected subset $\Sigma \subseteq S$ which has non-empty interior in the topological sense; and by $\text{cls}[\Sigma]$ we will denote the closure of Σ in S . Since S is either homeomorphic to $U(1)$ or to some interval via some homeomorphism κ with $\text{im}[\kappa] = S$, segments do not admit any isolated points. Then, for a subset $A \subseteq U(1)$, we let $\text{int}[A]$ denote its interior in the topological sense, and for a segment $\Sigma \subseteq S$, we define

$$\text{int}[\Sigma] := \kappa(\text{int}[\kappa^{-1}(\Sigma)]).$$

Obviously, then $\text{int}[\Sigma]$ is a segment as well; and $\text{int}[S]$ is the interior of S in the sense of manifolds with boundary. The elements of $\partial[\Sigma] := \text{cls}[\Sigma] - \text{int}[\Sigma]$ are called the **boundary points** of Σ ; and by a **boundary segment** of Σ , we will understand a segment Σ' with $\Sigma' \subset \Sigma$, such that $\Sigma - \Sigma'$ is a segment as well.

Now, each segment $\Sigma \subseteq S$ is a 1-dimensional embedded analytic submanifold with boundary of S . More precisely, for each $z \in \text{int}[\Sigma]$, we find an analytic chart (U, ψ) around z with $U \subseteq \text{int}[\Sigma] \subseteq \text{int}[S]$; and if z is a boundary point of Σ , we find an analytic chart (U, ψ) centred at z , such that $\psi(U \cap \Sigma) = [0, i) \subseteq \mathbb{R}_{\geq 0}$ holds. Such charts will be called **submanifold charts** of Σ in the following.

Then, if N is an analytic manifold with boundary, and $f: \Sigma \rightarrow N$ an analytic (immersive) map,

$$\phi \circ f \circ \psi^{-1}|_{\psi(U \cap \Sigma)}$$

is an analytic (immersive) curve if (U, ψ) is a submanifold chart of Σ , as well as (V, ϕ) a chart of N with $f(U) \subseteq V$. A homeomorphism $\rho: \Sigma \rightarrow \Sigma'$ is said to be analytic diffeomorphism iff ρ and its inverse are analytic immersive maps.

Finally, $\varphi: G \times M \rightarrow M$ will always denote a non-contractive Lie group action, i.e., a left action of a Lie group G on an analytic manifold M , such that

- i) The map $\varphi_g: M \rightarrow M$, $x \mapsto \varphi(g, x)$ is analytic for each $g \in G$.
- ii) If $x \notin C \subseteq M$ holds for $|C| \geq 2$, there exists a neighbourhood U of x with $g \cdot C \not\subseteq U$ for each $g \in G$.

To simplify the notations, we usually will write $g \cdot x$ instead of $\varphi(g, x)$; and by

$$G_x := \{g \in G \mid g \cdot x = x\},$$

we will denote the stabilizer of $x \in M$. Moreover, we define

$$G_C := \bigcap_{x \in C} G_x \quad \text{for} \quad \emptyset \neq C \subseteq M \quad \text{as well as} \quad G_T := G_{\iota(T)} \quad \text{for} \quad \emptyset \neq T \subseteq S.$$

2.2 Basic facts on analytic 1-manifolds

In the following, let (S, ι) denote some fixed analytic 1-submanifold of the analytic manifold M , and let $\varphi: G \times M \rightarrow M$ denote some fixed non-contractive Lie group action. Then, for segments $\Sigma, \Sigma' \subseteq S$, we will write

$$g \cdot \iota|_\Sigma \sim_\circ \iota|_{\Sigma'}$$

iff $g \cdot \iota(\mathcal{O}) = \iota(\mathcal{O}')$ holds for open segments $\mathcal{O}, \mathcal{O}' \subseteq \text{int}[S]$ with $\mathcal{O} \subseteq \Sigma$ and $\mathcal{O}' \subseteq \Sigma'$, on which ι is an embedding.

Then, $g \cdot \iota|_\mathcal{O} = \iota \circ \rho$ holds for $\rho := \iota^{-1} \circ (g \cdot \iota)|_\mathcal{O}$, which is an analytic diffeomorphism, because \mathcal{O} is an analytic manifold without boundary.⁶ Conversely, if $g \cdot \iota|_\Sigma = \iota \circ \rho$ holds for some analytic diffeomorphism $\rho: \Sigma \rightarrow \Sigma'$, we obviously have $g \cdot \iota|_\Sigma \sim_\circ \iota|_{\Sigma'}$. Finally, we observe that

$$g \cdot \iota|_{\text{cls}[\Sigma]} \sim_\circ \iota|_{\text{cls}[\Sigma]} \quad \text{for} \quad g \in G \quad \implies \quad g \cdot \iota|_\Sigma \sim_\circ \iota|_\Sigma \quad (4)$$

⁶In fact, around each $z \in \mathcal{O}$, we find an analytic submanifold chart (V, ϕ) of $W := (g \cdot \iota)(\mathcal{O}) = \iota(\mathcal{O}')$ around $(g \cdot \iota)(z)$. Then, for $\mathcal{U} \subseteq \mathcal{O}$ open with $z \in \mathcal{U}$ and $(g \cdot \iota)(\mathcal{U}) \subseteq V$, we have $\rho|_\mathcal{U} = (\phi \circ \iota)^{-1}|_{\phi(W \cap V)} \circ (\phi \circ (g \cdot \iota))|_\mathcal{U}$, which is obviously analytic immersive.

holds. This is clear if S is homeomorphic to an interval; and in the other case, one only has to think about the situation where $\Sigma = S - \{z\}$ holds for some $z \in S$.

Next, let us observe that

Lemma 2.1

Let $\gamma: I \rightarrow M$ and $\gamma': I' \rightarrow M$ be embedded analytic immersive curves with $\gamma(t_n) = \gamma'(t'_n)$ for all $n \in \mathbb{N}$, for sequences $I - \{t\} \supseteq \{t_n\}_{n \in \mathbb{N}} \rightarrow t \in I$ and $I' - \{t'\} \supseteq \{t'_n\}_{n \in \mathbb{N}} \rightarrow t' \in I'$. Then, $\gamma(J) = \gamma'(J')$ holds for some open intervals $J \subseteq I$ and $J' \subseteq I'$ containing t and t' , respectively.

PROOF: Choose an analytic submanifold chart (O, ϕ) of $\text{im}[\gamma]$ with $\gamma(t) \in O$, mapping $\text{im}[\gamma] \cap O$ into the x_1 -axis. Moreover, let $J' \subseteq I'$ be an open interval containing t' with $\gamma(J') \subseteq O$. Then, t' is an accumulation point of zeroes of $\phi^k \circ \gamma'|_{J'}$ for $k = 2, \dots, \dim[M]$, so that $\phi^k \circ \gamma'|_{J'} = 0$ holds for each such k , by analyticity. Thus, we have $\phi(\gamma'(J')) \subseteq \phi(\text{im}[\gamma] \cap O)$, hence $\gamma(J) = \gamma'(J')$ for $J := \gamma^{-1}(\gamma'(J'))$. ■

From this, we conclude that

Corollary 2.2

Let (U, ψ) and (U', ψ') be charts of S around x and x' , respectively, and assume that $g \cdot \iota(x_n) = \iota(x'_n)$ holds for all $n \in \mathbb{N}$, for some $g \in G$, as well as sequences $U - \{x\} \supseteq \{x_n\}_{n \in \mathbb{N}} \rightarrow x$ and $U' - \{x'\} \supseteq \{x'_n\}_{n \in \mathbb{N}} \rightarrow x'$. Moreover, let $\tilde{\gamma}_\psi$ and $\tilde{\gamma}_{\psi'}$ be analytic immersive extensions of γ_ψ and $\gamma_{\psi'}$, respectively.

Then, $g \cdot \tilde{\gamma}_\psi(J) = \tilde{\gamma}_{\psi'}(J')$ holds for open intervals J and J' with $t \in J$ and $t' \in J'$, on which $\tilde{\gamma}_\psi$ and $\tilde{\gamma}_{\psi'}$ are embeddings, respectively. In particular,

- If $x, x' \in \partial[S]$ or $x, x' \in \text{int}[S]$ holds, then $g \cdot \iota(K) = \iota(K')$ holds for connected compact neighbourhoods $K \subseteq U$ and $K' \subseteq U'$ of x and x' , respectively.
- If $x \in \partial[S]$ and $x' \in \text{int}[S]$ or $x \in \text{int}[S]$ and $x' \in \partial[S]$ holds, then $g \cdot \iota(K) = \iota(K')$ holds for compact segments $K \subseteq U$ and $K' \subseteq U'$ with $x \in \partial[K]$ and $x' \in \partial[K']$.

PROOF: Define $t := \psi(x)$ and $t' := \psi'(x')$, and choose open intervals I and I' with $t \in I$ and $t' \in I'$, such that $\gamma := g \cdot \tilde{\gamma}_\psi|_I$ and $\gamma' := \tilde{\gamma}_{\psi'}|_{I'}$ are embeddings. Then, $\gamma(J) = \gamma'(J')$ holds for open intervals J and J' with $t \in J$ and $t' \in J'$ by Lemma 2.1, which shows the first claim. The two points then are just clear from the homeomorphism property of $[\gamma|_J]^{-1} \circ \gamma'|_{J'}$. ■

Thus, we have

Lemma 2.3

If $g \cdot \iota(\Sigma) = \iota(\Sigma')$ holds for segments $\Sigma, \Sigma' \subseteq S$ on which ι is an embedding, then $\iota^{-1} \circ (g \cdot \iota|_\Sigma): \Sigma \rightarrow \Sigma'$ is an analytic diffeomorphism.

PROOF: Since, $g \cdot \iota|_\Sigma$ and $\iota|_{\Sigma'}$ are embeddings, $\rho := \iota^{-1} \circ (g \cdot \iota|_\Sigma)$ is a homeomorphism.

Then, for $z \in \Sigma$, we define $z' := \rho(z)$, and choose submanifold charts (U, ψ) of Σ and (U', ψ') of Σ' around z and z' , respectively. Then, by Corollary 2.2, we find open intervals J, J' with $\psi(z) \in J$, $\psi'(z') \in J'$, and $g \cdot \tilde{\gamma}_\psi(J) = \tilde{\gamma}_{\psi'}(J')$, such that $g \cdot \tilde{\gamma}_\psi|_J$ and $\tilde{\gamma}_{\psi'}|_{J'}$ are embeddings. Then, the claim is clear from

$$(\tilde{\gamma}_{\psi'})^{-1} \circ (g \cdot \tilde{\gamma}_\psi)|_{\psi(U \cap \Sigma)} = \psi' \circ \rho \circ \psi^{-1}|_{\psi(U \cap \Sigma)},$$

because $(\tilde{\gamma}_{\psi'})^{-1} \circ (g \cdot \tilde{\gamma}_\psi|_J)$ is an analytic diffeomorphism. ■

Moreover, the proof of Lemma 2.3 shows that

Lemma 2.4

Let $C, C' \subseteq S$ be segments with $C \cap C' \neq \emptyset$, such that $\iota^{-1} \circ (g \cdot \iota)|_C$ and $\iota^{-1} \circ (g \cdot \iota)|_{C'}$ are homeomorphisms to their images. Then, $\rho := \iota^{-1} \circ (g \cdot \iota)|_{C \cup C'}$ is an analytic diffeomorphism to its image if S is homeomorphic to an interval or if we find compact segments $K, K' \subset \text{int}[S]$, such that $C \cup C' \subseteq K$ and $\rho(C) \cup \rho(C') \subseteq K'$ holds.

PROOF: By the proof of Lemma 2.3, it suffices to show that ρ is a homeomorphism.

For this, define $\Sigma := C \cup C'$ and $\Sigma' := \text{im}[\rho]$, and observe that $\rho: \Sigma \rightarrow \Sigma'$ is well defined and bijective. Then, ρ is continuous, because $\rho|_C$ and $\rho|_{C'}$ are continuous, and because if $\{z_n\}_{n \in \mathbb{N}} \subseteq \Sigma$ converges to some $z \in \Sigma$, we have

$$\begin{aligned} \{z_n\}_{n \in \mathbb{N}} \subseteq C - C' &\implies z \in C, \\ \{z_n\}_{n \in \mathbb{N}} \subseteq C' - C &\implies z \in C'. \end{aligned}$$

Here, we have used that S is homeomorphic to an interval or that $C \cup C' \subseteq \mathcal{K} \subset \text{int}[S]$ holds for some compact segment \mathcal{K} . Then, the same arguments show that ρ^{-1} is continuous as well. ■

In addition to that, we conclude that

Lemma 2.5

If S is compact without boundary, then $g \cdot \iota \sim_\circ \iota$ implies $g \cdot \iota(S) \subseteq \iota(S)$, hence $g \cdot \iota(S) = \iota(S)$. Thus, if $\Sigma \subseteq S$ is a segment, then $g \cdot \iota|_\Sigma = \iota \circ \rho$ holds for the (unique) analytic diffeomorphism $\rho := \iota^{-1} \circ (g \cdot \iota|_\Sigma): \Sigma \rightarrow \rho(\Sigma)$.

PROOF: The second statement is clear from the first one and Lemma 2.3, because ι is an embedding by compactness of S . Now, for the first statement, let Z denote the set of all $z \in S$, for which we find an open neighbourhood $U_z \subseteq S$ of z , such that $g \cdot \iota(U_z) = \iota(\mathcal{U})$ holds for some open subset $\mathcal{U} \subseteq S$. Then, Z is open by definition, and non-empty by assumption. Thus, if we can show that Z is closed, the claim follows from connectedness of S .

For this, let $x \notin Z$ be contained in the closure of Z , and choose a chart (U, ψ) around x . Since $U' \cap Z \neq \emptyset$ holds for each neighbourhood $U' \subseteq U$ of x , we find a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq U \cap Z$ with $\lim_n x_n = x$. Since $x'_n := \iota^{-1}(g \cdot \iota(x_n)) \in S$ holds for each $n \in \mathbb{N}$, by compactness of S , we can assume that $\lim_n x'_n = x' \in S$ exists. Then, by injectivity of ι and $g \cdot \iota$, we must have $x'_n \neq x'$ for each $n \in \mathbb{N}$. We choose a chart (U', ψ') around x' , and pass to a subsequence, in order to assure that $\{x'_n\}_{n \in \mathbb{N}} \subseteq U'$ holds. Then, $x \in Z$ is clear from Corollary 2.2, so that the claim follows. ■

Next, let us clarify that

Lemma 2.6

Let N be an analytic manifold with boundary, $\Sigma \subseteq S$ a segment, and $\alpha, \alpha': \Sigma \rightarrow N$ analytic maps. Then, if $C \subseteq S$ is a segment contained in Σ , we have

$$\alpha|_C = \alpha'|_C \implies \alpha = \alpha'.$$

PROOF: Let Σ' denote the union of all segments $C' \subseteq S$ with $C \subseteq C' \subseteq \Sigma$, for which $\alpha|_{C'} = \alpha'|_{C'}$ holds. Then, Σ' is connected, as well as closed in Σ by continuity of α and α' .

Now, for each $z \in \Sigma'$, we find a submanifold chart (U, ψ) of Σ around z , as well as a chart (V, ϕ) of N , such that $\alpha(U), \alpha'(U) \subseteq V$ holds. Then, since Σ' admits no isolated points, $\psi(z)$ is an accumulation point of zeroes of

$$\phi^k \circ \alpha \circ \psi^{-1}|_{\psi(U \cap \Sigma)} - \phi^k \circ \alpha' \circ \psi^{-1}|_{\psi(U \cap \Sigma)} \quad \text{for} \quad k = 1, \dots, \dim[N],$$

so that $\alpha|_{U \cap \Sigma} = \alpha'|_{U \cap \Sigma}$ is clear from analyticity. Thus, Σ' is in addition open in Σ , with what $\Sigma' = \Sigma$ holds by connectedness of Σ . ■

In particular, we have

Corollary 2.7

Let $\rho, \rho': \Sigma \rightarrow \Sigma'$ be analytic diffeomorphisms for segments $\Sigma, \Sigma' \subseteq S$. Then, if $C \subseteq S$ is a segment contained in Σ , we have

$$\rho|_C = \rho'|_C \implies \rho = \rho'.$$

Moreover, if we define $\mathcal{O}(S) := \{g \in G \mid g \cdot \iota \sim_\circ \iota\}$, then

Corollary 2.8

For each $g \in G$, we have

$$g \in \mathcal{O}(S) \quad \implies \quad g^{-1} \cdot q \cdot g \in G_S \quad \forall q \in G_S. \quad (5)$$

PROOF: Let $q \in G_S$, and $\mathcal{O}, \mathcal{O}' \subseteq \text{int}[S]$ be open and connected with $g \cdot \iota(\mathcal{O}) = \iota(\mathcal{O}')$. Then, we have

$$q \cdot (g \cdot \iota(z)) = g \cdot \iota(z) \quad \forall z \in \mathcal{O} \quad \implies \quad (g^{-1} \cdot q \cdot g) \cdot \iota|_{\mathcal{O}} = \iota|_{\mathcal{O}} \quad \xrightarrow{\text{Lemma 2.6}} \quad g^{-1} \cdot q \cdot g \in G_S. \quad \blacksquare$$

Finally, let us say that an analytic diffeomorphism $\rho: \Sigma \rightarrow \Sigma'$ between segment $\Sigma, \Sigma' \subseteq S$ is **maximal** iff there exists no proper extension of ρ , i.e., no analytic diffeomorphism $\tilde{\rho}: C \rightarrow C' \subseteq S$ with $\rho = \tilde{\rho}|_{\Sigma}$, for $C \subseteq S$ a segment properly containing Σ . Then,

Lemma and Definition 2.9 (Maximal extension)

If S is homeomorphic to an interval, each analytic diffeomorphism $\rho: \Sigma \rightarrow \Sigma'$ admits a unique maximal extension, denoted by $\bar{\rho}$ in the following.

PROOF: Let E denote the set of all analytic diffeomorphisms extending ρ , and define

$$\bar{\rho}: \bigcup_{\tau \in E} \text{dom}[\tau] =: \bar{\Sigma} \rightarrow \text{im}[\bar{\rho}] \subseteq S \quad \text{by} \quad \bar{\rho}(x) := \tau(x) \quad \text{for} \quad \tau \in E \quad \text{with} \quad x \in \text{dom}[\tau].$$

Obviously, $\bar{\Sigma} \subseteq S$ is a segment; and $\bar{\rho}$ is well defined by Corollary 2.7, because if $\tau' \in E$ is another extension with $x \in \text{dom}[\tau']$, then $\Sigma \subseteq \text{dom}[\tau] \cap \text{dom}[\tau'] =: C \ni x$ holds, whereby C is connected as S is homeomorphic to an interval. Finally, $\bar{\rho}$ is maximal, because for each $\xi \in E$, we have $\text{dom}[\xi] \subseteq \bar{\Sigma}$ by definition. In particular, if $\xi \in E$ is maximal, we must have $\text{dom}[\xi] = \bar{\Sigma}$, hence $\xi = \bar{\rho}$ by Corollary 2.7. \blacksquare

Proposition 2.10

Suppose that S is homeomorphic to an interval, and let $\rho: \mathcal{O} \rightarrow \mathcal{O}'$ be an analytic diffeomorphism for open segments $\mathcal{O}, \mathcal{O}' \subseteq S$, such that $g \cdot \iota|_{\mathcal{O}} = \iota \circ \rho$ holds for some $g \in G$.

If $\Sigma \subseteq S$ is a segment with $\mathcal{O} \subseteq \Sigma$, then $C := \Sigma \cap \text{dom}[\bar{\rho}]$ is a segment, and $g \cdot \iota|_C = \iota \circ \bar{\rho}|_C$ holds by Corollary 2.7. Here, $C \subset \Sigma$ implies connectedness of $S - \bar{\rho}(C)$, whereby $S = \bar{\rho}(C)$ holds if $\Sigma - C$ is not connected.

PROOF: First observe that C is a segment, because $\text{dom}[\bar{\rho}]$ and Σ are connected and contain \mathcal{O} , and because S is homeomorphic to an interval D . Let $\kappa: D \rightarrow S$ denote the corresponding homeomorphism, and define $B := \kappa^{-1}(C)$, as well as $B' := \kappa^{-1}(\bar{\rho}(C))$.

Then, $C \subset \Sigma$ means that B is properly contained in $\kappa^{-1}(\Sigma)$; and then we find a boundary point $x \in \Sigma$ of C , and a chart (U, ψ) around x , such that $\mathcal{U} \cap C \neq \emptyset$ and $\mathcal{U} \cap (\Sigma - C) \neq \emptyset$ holds for each neighbourhood $\mathcal{U} \subseteq U$ of x . Moreover, we find $\{x_n\}_{n \in \mathbb{N}} \subseteq C - \{x\}$ with $\lim_n x_n = x$, such that $\{\kappa^{-1}(x_n)\}_{n \in \mathbb{N}}$ is monotonously decreasing or increasing. Then, $\{(\kappa^{-1} \circ \bar{\rho})(x_n)\}_{n \in \mathbb{N}} \subseteq B'$ is monotonously increasing or decreasing as well, because $\kappa^{-1} \circ \bar{\rho} \circ \kappa|_B: B \rightarrow B'$ is a homeomorphism. Let us define $x'_n := \bar{\rho}(x_n)$ for each $n \in \mathbb{N}$.

Then, if $S - \bar{\rho}(C)$ is not connected, $D - B'$ is not connected, so that $\lim_n \kappa^{-1}(x'_n) \in D$ must exist by monotonicity. Thus, $\lim_n x'_n = x'$ holds for some $x' \in S$, and by construction, we have $\{x'_n\}_{n \in \mathbb{N}} \subseteq S - \{x'\}$ as well as $g \cdot \iota(x_n) = (\iota \circ \bar{\rho})(x_n) = \iota(x'_n)$ for each $n \in \mathbb{N}$. We choose a chart (U', ψ') around x' , and pass to a subsequence, in order to achieve that $\{x'_n\}_{n \in \mathbb{N}} \subseteq U'$ holds.

Then, Corollary 2.2 shows that $g \cdot \iota(\mathcal{K}) = \iota(\mathcal{K}')$ holds for connected compact segments $\mathcal{K}, \mathcal{K}'$ with $x \in \mathcal{K}$ and $x' \in \mathcal{K}'$, such that

- \mathcal{K} is a neighbourhood of x if $x \in \partial[S]$ or $x, x' \in \text{int}[S]$ holds. In this case, $\iota^{-1} \circ (g \cdot \iota)|_{\mathcal{K} \cup C}$ is an analytic diffeomorphism to its image by Lemma 2.4, so that $\bar{\rho}$ must be defined on $\mathcal{K} \cup C$ by maximality. This, however, contradicts the definition of C , because $\mathcal{K} \cap (\Sigma - C) \neq \emptyset$ holds.
- In the other case, we have $x \in \text{int}[S]$ and $x' \in \partial[S]$, hence $x' \notin \bar{\rho}(C)$, because otherwise $S - \bar{\rho}(C)$, i.e., $D - B'$ would be connected. Thus, $x \notin C$ must hold by continuity of $\bar{\rho}$, so that we obtain a contradiction to maximality of $\bar{\rho}$, just by the same arguments as above.

Finally, if $\Sigma - C$ is not connected, we even find two boundary points x^\pm of C as above, whereby we can assume that $\kappa^{-1}(x^-) < \kappa^{-1}(x^+)$ holds. Obviously, then we find sequences $\{x_n^\pm\}_{n \in \mathbb{N}} \subseteq C - \{x^\pm\}$ with $\lim_n x_n^\pm = x^\pm$, such that $\{\kappa^{-1}(x_n^-)\}_{n \in \mathbb{N}}$ monotonously decreasing, and $\{\kappa^{-1}(x_n^+)\}_{n \in \mathbb{N}}$ is monotonously increasing. Thus, if $\bar{\rho}(C)$ is properly contained in S , the image under $\kappa^{-1} \circ \bar{\rho}$ of one of these sequences must converge to some $t \in D$. But, then we can choose charts around $x := \rho^{-1}(\kappa(t)) \in \{x^-, x^+\}$ and $x' := \kappa(t)$, in order to derive a contradiction to maximality of $\bar{\rho}$, just by the same arguments we have already used above. ■

2.3 Free segments

Let us say that (S, ι) is **free** iff it admits a **free segment**, i.e., a segment $\Sigma \subseteq S$ with

$$g \cdot \iota|_\Sigma \sim_\circ \iota|_\Sigma \quad \text{for } g \in G \quad \implies \quad g \cdot \iota|_\Sigma = \iota|_\Sigma \quad \iff \quad g \in G_S;$$

and observe that then each segment $\Sigma' \subseteq S$ with $\Sigma' \subseteq \Sigma$, is a free segment as well. We will say that a free segment $\Sigma \subseteq S$ is maximal iff $\Sigma \subseteq \Sigma'$ implies $\Sigma = \Sigma'$ for each free segment $\Sigma' \subseteq S$. Then, Σ is closed in S by (4), and we have

Lemma 2.11

If $\Sigma \subseteq S$ is a free segment, then we find $\Sigma' \subseteq S$ maximal with $\Sigma \subseteq \Sigma'$.

PROOF: Let \mathfrak{D} denote the set of all free segments $C \subseteq S$ which contain Σ . We order \mathfrak{D} by inclusion, and observe that then each chain \mathfrak{C} in \mathfrak{D} has the upper bound $B := \bigcup_{C \in \mathfrak{C}} C$. In fact, B is obviously a segment; and it is free, because

$$g \cdot \iota|_B \sim_\circ \iota|_B \quad \implies \quad g \cdot \iota|_C \sim_\circ \iota|_C \quad \text{for some } C \in \mathfrak{C} \quad \implies \quad g \in G_S, \quad (6)$$

with what the set of maximal elements in \mathfrak{D} is non-empty by Zorn's lemma. For the first implication in (6),

- ▷ Let $g \cdot \iota(\mathcal{O}) = \iota(\mathcal{O}')$ hold for open segments $\mathcal{O}, \mathcal{O}' \subseteq B \cap \text{int}[S]$, on which ι is an embedding.
- ▷ Since $C \cap \mathcal{O} \neq \emptyset$ holds for some $C \in \mathfrak{C}$, we find open segments $\mathcal{U} \subseteq C \cap \mathcal{O}$ and $\mathcal{U}' \subseteq \mathcal{O}'$ with $g \cdot \iota(\mathcal{U}) = \iota(\mathcal{U}')$.
For this observe that $C \cap \mathcal{O}$ has non-empty interior (S is either homeomorphic to $U(1)$ or to an interval), and that $\iota^{-1} \circ g \cdot \iota|_{\mathcal{O}} : \mathcal{O} \rightarrow \mathcal{O}'$ is a homeomorphism.
- ▷ Then, $C' \cap \mathcal{U}' \neq \emptyset$ holds for some $C' \in \mathfrak{D}$ with $C \subseteq C'$, hence $\mathcal{U} \subseteq C'$; so that the same arguments as in the previous point show that $g \cdot \iota|_{C'} \sim_\circ \iota|_{C'}$ holds. ■

Moreover,

Lemma 2.12

Let $\Sigma, \Sigma' \subseteq S$ be segments with $g \cdot \iota|_\Sigma = \iota \circ \rho$ for some $g \in G$, and some analytic diffeomorphism $\rho : \Sigma \rightarrow \Sigma'$. Then, Σ' is free if Σ is free; and if Σ, Σ' are compact with $\Sigma \subseteq \text{int}[S]$, then Σ' is maximal if Σ is maximal.

PROOF: The first statement is clear, because for each $q \in G$, we have

$$\begin{aligned} q \cdot \iota|_{\Sigma'} \sim_\circ \iota|_{\Sigma'} &\implies q \cdot \iota|_{\Sigma'} \circ \rho \sim_\circ \iota|_{\Sigma'} \circ \rho &\implies q \cdot (g \cdot \iota|_\Sigma) \sim_\circ (g \cdot \iota|_\Sigma) \\ &\implies (g^{-1} \cdot q \cdot g) \cdot \iota|_\Sigma \sim_\circ \iota|_\Sigma &\implies g^{-1} \cdot q \cdot g \in G_S \\ &\implies (q \cdot g) \cdot \iota|_\Sigma = g \cdot \iota|_\Sigma &\implies q \cdot (g \cdot \iota|_\Sigma \circ \rho^{-1}) = g \cdot \iota|_\Sigma \circ \rho^{-1} \\ &\implies q \cdot \iota|_{\Sigma'} = \iota|_{\Sigma'}. \end{aligned}$$

The second statement is clear if $\Sigma = \text{int}[S]$ holds, because then $S = \Sigma$ is compact without boundary, so that $\Sigma' = S$ holds by Lemma 2.5.

Now, $\Sigma \subset \text{int}[S]$ holds, and if $\Sigma'' \subseteq S$ is a free segment with $\Sigma' \subset \Sigma''$, then we find a boundary point $x \in \text{int}[S]$ of Σ , such that $x' := \rho(x) \in \text{int}[\Sigma'']$ is a boundary point of Σ' . Then, we find charts (U, ψ) and (U', ψ') around x and x' , respectively, such that, for the case that S is compact without boundary, additionally $U \cup \Sigma \subseteq \mathcal{K}$ and $U' \cup \Sigma' \subseteq \mathcal{K}'$ holds for compact segments $\mathcal{K}, \mathcal{K}' \subset \text{int}[S]$. Then, by Corollary 2.2, we find connected neighbourhoods $C \subseteq U \subseteq \text{int}[S]$ of x , and $C' \subset \Sigma'' \cap U' \subseteq \text{int}[S]$ of x' , such that $\iota^{-1} \circ (g \cdot \iota)|_C : C \rightarrow C'$ is a homeomorphism. Thus, Lemma 2.4 shows that $\tilde{\rho} := \iota^{-1} \circ (g \cdot \iota)|_{C \cup \Sigma}$ is an analytic diffeomorphism, so that the first part of this lemma, applied to $\tilde{\rho}^{-1}$ and the free segment $\Sigma' \subset C' \cup \Sigma' \subset \Sigma''$, shows that Σ is not maximal. This, however, contradicts the assumptions. ■

Finally, let us observe that (D, γ) is an embedded analytic 1-submanifold of M if $\gamma: D \rightarrow M$ is an embedded analytic curve.⁷ Then, $D' \subseteq D$ is a free segment iff it is a free interval in the sense of Definition 4.1 in [1], see also Definition 3.5 in [1], with what Lemma 4.10 in [1] shows that

Lemma 2.13

Let $\gamma: [t', t] \rightarrow M$ be an embedded analytic curve, and $[a', a]$ a free segment. Then,

$$\begin{aligned} a < t &\implies [a, k] \text{ is a free segment for some } a < k \leq t, \\ t' < a' &\implies [k', a'] \text{ is a free segment for some } t' \leq k' < a'. \end{aligned}$$

In particular, if Σ is a free segment with boundary point $z \in \text{int}[S]$, then we find a free segment $\Sigma' \subseteq S$ with $\Sigma \cap \Sigma' = \{z\}$. This is just clear from the above lemma, applied to γ_ψ for a suitable chart (U, ψ) around z .

3 Decompositions

In the following, let M denote some fixed analytic manifold with non-contractive Lie group action $\varphi: G \times M \rightarrow M$. Moreover, let (S, ι) be some fixed free analytic 1-submanifold of M such that S is not a free segment by itself. In this section, we are going to show that S naturally decomposes into maximal free segments, mutually and uniquely related by the group action. Here, several special cases have to be worked out, basically subdividing into the situations where S admits a compact maximal segment $\Sigma \subset \text{int}[S]$ or not. More precisely, if S is compact without boundary, the first situation must hold, because each maximal segment Σ is necessarily compact as it is closed; and then S is build up finitely many translates of Σ . If S is homeomorphic to an interval with $\Sigma \subset \text{int}[S]$ maximal and compact, in general countably many translates are necessary; and if no such Σ exists, S decomposes exactly into two maximal free segment, pinned together at their common boundary point.

3.1 Basic facts

Assume that $\Sigma \subseteq S$ is a free segment, and that we are given $g \in G - G_S$ such that $g \cdot \iota|_C = \iota \circ \rho$ holds for some analytic diffeomorphism $\rho: C \rightarrow C'$, for some segment $C \subseteq \Sigma$. Then, $\Sigma \cap C'$ can only contain the boundary points of Σ , because otherwise we find an open segment $\mathcal{O}' \subseteq \Sigma \cap C'$, and conclude for $\Sigma \supseteq \mathcal{O} := \rho^{-1}(\mathcal{O}')$

$$g \cdot \iota|_{\mathcal{O}} = \iota \circ \rho|_{\mathcal{O}} \implies g \cdot \iota|_{\Sigma} \sim_{\circ} \iota|_{\Sigma} \implies g \in G_S.$$

In particular, S cannot be contained in $C' = \rho(C)$, and Σ must be properly contained in S .

Now, let us assume that ρ is defined on an open segment $\mathcal{O} \subseteq \Sigma$. Then,

- a) If $\kappa: D \rightarrow S$ is a homeomorphism, by Proposition 2.10, we have $g \cdot \iota|_C = \iota \circ \bar{\rho}|_C$ for $C = \Sigma \cap \text{dom}[\bar{\rho}]$, whereby $\Sigma - C$ must be connected, since otherwise $S = \bar{\rho}(C)$ holds.

Thus, we either have $C = \Sigma$ or C is a boundary segment of Σ . In the second case, $S - \bar{\rho}(C)$ is connected by Proposition 2.10, so that both $\bar{\rho}(C)$ and $S - \bar{\rho}(C)$ are boundary segments of S . In any case, $\iota|_{\bar{\rho}(C)} = g \cdot \iota|_C \circ \bar{\rho}^{-1}|_{\bar{\rho}(C)}$ is an embedding, and $\bar{\rho}(C)$ is compact if C is so.

Finally, $\Sigma \cap C'$ can only contain the boundary points of Σ , and since S is homeomorphic to an interval, it even must be empty or singleton.

- b) If S is compact without boundary, then $g \cdot \iota|_{\Sigma} = \iota \circ \bar{\rho}|_{\Sigma}$ holds by Lemma 2.5. Moreover, $\Sigma' := \bar{\rho}(\Sigma)$ is compact as Σ is compact, and $\Sigma \cap \Sigma'$ can only contain the boundary points of Σ . Then, Lemma 2.12 shows that $\Sigma' := \bar{\rho}(\Sigma)$ is free; and even maximal if Σ is maximal.

Now, if $\Sigma \subset \text{int}[S]$ is a compact segment, and Σ_b is a boundary segment of Σ , in the following, we will mean that b is the boundary point of Σ that is contained in Σ_b . We define $G := G/G_S$, and obtain the following analogue to Proposition 4.11 in [1].

⁷More precisely, this means that $\gamma = \tilde{\gamma}|_D$ holds for some embedded analytic immersive curve $\tilde{\gamma}: I \rightarrow M$.

Proposition 3.1

Let $\Sigma \subset S$ be a compact maximal segment, and $z \in \text{int}[S]$ a boundary point of Σ . Then, there exists a unique class $[g] \neq [e]$, a compact free segment $\Sigma_z \subseteq S$ with $\{z\} = \Sigma \cap \Sigma_z$, and a compact boundary segment Σ_b of Σ , such that $g \cdot \iota(\Sigma_b) = \iota(\Sigma_z)$ as well as $g \cdot \iota(b) = \iota(z)$ holds.

PROOF: Replacing S by a suitable segment containing Σ if necessary, we can assume that S is homeomorphic to an interval via $\kappa: D \rightarrow S$, with $\kappa^{-1}(\Sigma) = [k', k]$ for $k = \kappa^{-1}(z) \in \text{int}[D]$.

Then, for uniqueness of $[g]$, let us assume that the statement also holds for $[g'] \neq [e]$ w.r.t. Σ'_z and $\Sigma_{b'}$. Then, $\Sigma_z \cap \Sigma'_z$ is a compact segment contained in $\iota^{-1}(g \cdot \iota(\Sigma_b)) \cap \iota^{-1}(g' \cdot \iota(\Sigma_{b'}))$, so that Lemma 2.3 shows

$$g' \cdot \iota|_{\Sigma_{b'}} \sim_{\circ} g \cdot \iota|_{\Sigma_b} \implies g^{-1}g' \cdot \iota|_{\Sigma} \sim_{\circ} \iota|_{\Sigma} \implies [g] = [g'].$$

Now, for existence, let us choose (Lemma 2.13) free segments $\{\Sigma'_n\}_{n \in \mathbb{N}}$ with $\Sigma'_{n+1} \subseteq \Sigma'_n$ and $\Sigma \cap \Sigma'_n = \{z\}$ for all $n \in \mathbb{N}$, such that $\{z\} = \bigcap_{n \in \mathbb{N}} \Sigma'_n$ holds. Then, $\Sigma_n := \Sigma \cup \Sigma'_n$ is not a free segment for each $n \in \mathbb{N}$, by maximality of Σ . Thus, for each $n \in \mathbb{N}$, we find $g_n \in G - G_S$ with $g_n \cdot \iota|_{\Sigma_n} \sim_{\circ} \iota|_{\Sigma_n}$; and

- ▷ Since Σ and Σ'_n are free segments, we must have $g_n \cdot \iota|_{\Sigma} \sim_{\circ} \iota|_{\Sigma'_n}$ or $g_n \cdot \iota|_{\Sigma'_n} \sim_{\circ} \iota|_{\Sigma}$.
- ▷ Thus, replacing g_n by g_n^{-1} if necessary, we can assume that for each $n \in \mathbb{N}$, we have

$$g_n \cdot \iota|_{\Sigma} \sim_{\circ} \iota|_{\Sigma'_n} \implies g_n \cdot (\iota|_{\Sigma})|_{\mathcal{O}_n} = \iota \circ \rho_n \quad (7)$$

for some analytic diffeomorphism $\rho_n: \Sigma \supset \mathcal{O}_n \rightarrow \mathcal{O}'_n \subset \Sigma'_n$.

Then, if $\bigcup_{n \in \mathbb{N}} \{[g_n]\} \subseteq G - \{[e]\}$ is finite, passing to a subsequence, we can assume that $[g_n] = [g]$ holds for each $n \in \mathbb{N}$, for some $[e] \neq [g] \in G$. Then, we find sequences $\{x_n\}_{n \in \mathbb{N}} \subseteq \Sigma$ and $\{x'_n\}_{n \in \mathbb{N}} \subseteq \Sigma'_0 - \{z\}$ with

$$x'_n \in \Sigma'_n \quad \text{and} \quad g \cdot \iota(x_n) = g_n \cdot \iota(x_n) = \iota(x'_n) \quad \forall n \in \mathbb{N}.$$

Now, by the choice of Σ'_n , we necessarily have $\lim_n x'_n = z$; and by compactness of Σ , we can pass to a subsequence in order to assure that $\lim_n x_n = x \in \Sigma$ exist as well. Then, we have $\{x_n\}_{n \in \mathbb{N}} \subseteq \Sigma - \{x\}$ by injectivity of $g \cdot \iota$ and ι , and because $g \cdot \iota(x) = \iota(z)$ holds by continuity. Thus, Corollary 2.2 shows that $g \cdot \iota(\mathcal{K}) = \iota(\mathcal{K}')$ holds for compact segments \mathcal{K} and \mathcal{K}' containing x and z , respectively, whereby \mathcal{K} is a neighbourhood of x , since $z \in \text{int}[S]$ holds. Then, $b := x$ must be a boundary point of Σ , since otherwise $g \cdot \iota|_{\Sigma} \sim_{\circ} \iota|_{\Sigma}$ holds. Thus, the claim holds for $\Sigma_b := \mathcal{K} \cap \Sigma$ and $\Sigma_z := \rho(\Sigma_b)$, for $\rho := \iota^{-1} \circ (g \cdot \iota)|_{\Sigma_b}$, just by the discussions in the beginning of this subsection.

Now, if $\bigcup_{n \in \mathbb{N}} \{[g_n]\} \subseteq G - \{[e]\}$ is not finite, passing to a subsequence, we can assume that $[g_n] \neq [g_m]$ holds for each $n \neq m$. We define

$$C_n := \Sigma \cap \text{dom}[\bar{\rho}_n] \quad \text{and} \quad C'_n := \bar{\rho}_n(C_n) \subseteq S \quad \forall n \in \mathbb{N}, \quad (8)$$

so that $g_n \cdot (\iota|_{\Sigma})|_{C_n} = \iota \circ \bar{\rho}_n|_{C_n}$ holds for all $n \in \mathbb{N}$. Then, $C'_n \cap C'_m$ must have empty interior for $m \neq n$, because otherwise $g_n \cdot \iota|_{\Sigma} \sim_{\circ} g_m \cdot \iota|_{\Sigma}$, hence $[g_n] = [g_m]$ holds, which contradicts the choices. By the same argument, $\Sigma \cap C'_n$ must have empty interior for each $n \in \mathbb{N}$, because $[g_n] \neq [e]$ holds.

Now, let $K, K_n, D_n \subseteq D$ denote the intervals, for which

$$\kappa(K) = \Sigma \quad \text{as well as} \quad \kappa(K_n) = \Sigma_n \quad \text{and} \quad \kappa(D_n) = C'_n$$

holds for each $n \in \mathbb{N}$. Then, we have $K = [k', k]$, as well as $K_n = [k', k + \Delta_n]$ for $\Delta_n > 0$ with $\Delta_{n+1} < \Delta_n$ for all $n \in \mathbb{N}$, and $\lim_n \Delta_n = 0$. Here, for each $n \in \mathbb{N}$, we have $D_n \subseteq D \cap [k, \infty)$ by (7), and even $D_n \subseteq D \cap [k + \epsilon_n, \infty)$ for some $\epsilon_n > 0$.

In fact, otherwise, we find $m > n$, such that $D_m \cap D_n$ has non-empty interior, which contradicts that $C'_m \cap C'_n$ has empty interior for such m and n . For this, just let $m > n$ be such large that $(k, \Delta_m] \subseteq D_n$ holds, and observe that $D_m \cap (k, \Delta_m]$ has non-empty interior by (7).

Then, for each $n \in \mathbb{N}$, we find $m > n$ with $\Delta_m \leq \epsilon_n$, hence $K \leq D_m \leq D_n$, whereby $B \leq B'$ for two intervals B, B' just means that $\sup(B) \leq \inf(B')$ holds. Thus, passing to a subsequence, we can assume that

$K \leq D_{n+1} \leq D_n$ holds for all $n \in \mathbb{N}$. Then, we must have $C_n = \Sigma$ for each $n \geq 1$, since otherwise $S - C'_n$ is connected by Proposition 2.10, implying that Σ or C'_0 is contained in C'_n , because we have $\Sigma \leq C'_n \leq C'_0$.

Thus, for each $\epsilon > 0$, we find $n \geq 1$ with $D_n \subseteq (k, k + \epsilon)$, so that for each neighbourhood $U \subseteq M$ of $z = \kappa(k)$, we find $n \geq 1$ with

$$g_n \cdot \iota(\Sigma) = g_n \cdot \iota(C_n) = \iota(C'_n) = (\iota \circ \kappa)(D_n) \subseteq U,$$

which contradicts Property ii) of φ . ■

3.2 Compact maximal segments

Let $\Sigma \subset S$ be a compact maximal segment with boundary points z_{\pm} . Moreover, assume that $z \in \{z_-, z_+\}$ is contained in $\text{int}[S]$, and let $[g]$, Σ_b , and Σ_z have the same meaning as in Proposition 3.1. In this situation, we will say that Σ is **positive** or **negative** iff $g \notin G_z$ or $g \in G_z$ holds, respectively. Figuratively speaking,

- positivity means that g shifts $\iota(\Sigma_b)$ in such a way that $\iota(b) \neq \iota(z)$ is mapped to $\iota(z)$,
- negativity means that g flips $\iota(\Sigma_b)$ at $\iota(b) = \iota(z)$.

Now, let us first show that

$$g \in G_z \quad \implies \quad [g] = [g^{-1}]. \quad (9)$$

In fact, we have $g \cdot \iota(\Sigma_b) = \iota(\Sigma_z)$ for $b = z$, hence $g \cdot \iota|_{\Sigma_b} = \iota \circ \rho|_{\Sigma_b}$ for the analytic diffeomorphism $\rho := \iota^{-1} \circ (g \cdot \iota|_{\Sigma_b})$ by Lemma 2.3. Then, fixing a chart (U, ψ) around z with $U \subseteq \Sigma_b \cup \Sigma_z$, by Corollary 2.2, we find connected compact neighbourhoods $\mathcal{K}, \mathcal{K}' \subseteq U$ of z , for which $g \cdot \iota(\mathcal{K}) = \iota(\mathcal{K}')$ holds. Then, we have

$$\begin{aligned} g \cdot \iota(\mathcal{K} \cap \Sigma_b) = (\mathcal{K}' \cap \Sigma_z) & \xrightarrow{b=z} g \cdot \iota(\mathcal{K} \cap \Sigma_z) = \iota(\mathcal{K}' \cap \Sigma_b) \\ & \implies g^{-1} \cdot \iota(\mathcal{K}' \cap \Sigma_b) = \iota(\mathcal{K} \cap \Sigma_z), \end{aligned}$$

so that the uniqueness statement in Proposition 3.1 shows the claim. Next, let us show that

Lemma 3.2

A compact maximal segment $\Sigma \subset \text{int}[S]$ is either positive or negative.

For this, let $z_{\pm} \in \text{int}[S]$, and denote by $[g_{\pm}]$, $\Sigma_{z_{\pm}}$, and $\Sigma_{b_{\pm}}$ the respective classes, free segments, and compact boundary segments from Proposition 3.1, for which

$$g_{\pm} \cdot \iota(\Sigma_{b_{\pm}}) = \iota(\Sigma_{z_{\pm}}) \quad \text{holds with} \quad g_{\pm} \cdot \iota(b_{\pm}) = \iota(z_{\pm}) \quad \text{and} \quad \{z_{\pm}\} = \Sigma \cap \Sigma_{z_{\pm}}$$

for $b_{\pm} \in \{z_-, z_+\}$. Then, in order to prove Lemma 3.2, it suffices to show that

$$g_+ \notin G_{z_+} \implies [g_-] = [g_+^{-1}] \quad \text{as well as} \quad g_- \notin G_{z_-} \implies [g_+] = [g_-^{-1}], \quad (10)$$

because

- $g_+ \notin G_{z_+}$ implies $b_+ = z_-$, i.e., $g_+ \cdot \iota(z_-) = \iota(z_+)$, hence $g_+^{-1} \notin G_{z_-}$,
- $g_- \notin G_{z_-}$ implies $b_- = z_+$, i.e., $g_- \cdot \iota(z_+) = \iota(z_-)$, hence $g_-^{-1} \notin G_{z_+}$.

Now, for (10), assume that $g_+ \notin G_{z_+}$ holds (the right hand side of (10) follows analogously). Then, we have $g_+^{-1} \cdot \iota(\Sigma_{z_+}) = \iota(\Sigma_{b_+})$ for $b_+ = z_-$, and choose charts (U_{\pm}, ψ_{\pm}) around z_{\pm} with

$$U_+ \subseteq \Sigma \cup \Sigma_{z_+} \quad \text{as well as} \quad U_- \subseteq \Sigma_{z_-} \cup \Sigma_{b_+}. \quad (11)$$

▷ We conclude from Corollary 2.2 that $g_+^{-1} \cdot \iota(\mathcal{K}_+) = \iota(\mathcal{K}_-)$ holds for connected compact neighbourhoods $\mathcal{K}_+ \subseteq U_+$ and $\mathcal{K}_- \subseteq U_-$ of z_+ and z_- , respectively.

▷ Since $z_{\pm} \in \text{int}[S]$ holds, we have

$$g_+^{-1} \cdot \iota(\mathcal{K}_+ \cap \Sigma_{z_+}) = \iota(\mathcal{K}_- \cap \Sigma_{b_+}) \stackrel{(11)}{\implies} g_+^{-1} \cdot \iota(\mathcal{K}_+ \cap \Sigma) = \iota(\mathcal{K}_- \cap \Sigma_{z_-}),$$

so that the uniqueness statement in Proposition 3.1 shows the claim.

The above arguments now also show that

Lemma 3.3

If $S' \subseteq S$ is a segment and Σ positive w.r.t. S' , then Σ is compact maximal w.r.t. S .

PROOF: By definition, some boundary point z_+ of Σ is contained in $\text{int}[S'] \subseteq \text{int}[S]$; and we denote the unique class from Proposition 3.1 (applied to S' , Σ , and z_+) by $[g_+]$. Then, for each neighbourhood U of z_+ , obviously $g_+ \cdot \iota|_{\Sigma} \sim_{\circ} \iota|_U$ holds, so that z_+ must be a boundary point of each free segment $\Sigma' \subseteq S$ with $\Sigma \subseteq \Sigma'$.

Similarly, if $z_- \in \text{int}[S]$ holds for the other boundary point of Σ , for each neighbourhood $U \subseteq \text{int}[S]$ of z_- , we have $g_+^{-1} \cdot \iota|_{\Sigma} \sim_{\circ} \iota|_U$, just by the same arguments, we have used to prove (10). Thus, z_- must be a boundary point of each free segment Σ' containing Σ as well, so that $\Sigma \subseteq \Sigma'$ implies $\Sigma = \Sigma'$. ■

Finally, let us observe that

$$[g_k] = [g'_k] \quad \text{for } k = 1, \dots, n \quad \implies \quad [g_n \cdot \dots \cdot g_1] = [g'_n \cdot \dots \cdot g'_1], \quad (12)$$

provided that $g_k \cdot \dots \cdot g_1 \in \mathcal{O}(S)$ holds for $k = 1, \dots, n-1$, which just follows inductively from Corollary 2.8.

3.2.1 Compact without boundary

If S is compact without boundary, each maximal segment $\Sigma \subseteq S$ is necessarily compact as it is closed by (4); and we define

Definition 3.4 (Σ -decomposition)

If S is compact without boundary, and $\Sigma \subset S$ is maximal, a Σ -decomposition of S is a collection of compact maximal segments $\Sigma_0, \dots, \Sigma_n$, and classes $[g_0], \dots, [g_n] \in \mathbf{G}$ with $\Sigma_0 = \Sigma$, $S = \bigcup_{k=0}^n \Sigma_k$, as well as $g_k \cdot \iota(\Sigma_0) = \iota(\Sigma_k)$ for each $0 \leq k \leq n$, such that

- If $n = 1$, then $\Sigma_0 \cap \Sigma_1 = \{z_-, z_+\}$ consists of the two boundary points of $\Sigma_0 = \Sigma$.
- If $n \geq 2$, then $\Sigma_p \cap \Sigma_q$ is singleton for $|p - q| \in \{1, n\}$, and empty otherwise.

The positive integer n will be called the **length** of the decomposition in the following.

Then,

Remark 3.5

In the situation of Definition 3.4,

- 1) The classes $[g_0], \dots, [g_n]$ are necessarily mutually different, because $[g_p] = [g_q]$ for some $0 \leq p, q \leq n$, implies $\Sigma_p = \Sigma_q$ by injectivity of ι , hence $p = q$ by the last two points in Definition 3.4.
- 2) We have $[g_0] = [e]$, because $g_0 \cdot \iota(\Sigma_0) = \iota(\Sigma_0)$ implies $g_0 \cdot \iota|_{\Sigma_0} \sim_{\circ} \iota|_{\Sigma_0}$ by compactness of Σ_0 .
- 3) If S is of length 1, then $[g_1] = [g_1^{-1}]$ holds. This is clear from (9) if Σ is negative, and from (10) if Σ is positive.
- 4) Let $\Sigma' = \Sigma_p$ hold for some $0 \leq p \leq n$. Then, $\Sigma'_0, \dots, \Sigma'_n$ and $[g'_0], \dots, [g'_n]$, defined by

$$\Sigma'_k := \Sigma_{\zeta(k)} \quad \text{and} \quad [g'_k] := [g_{\zeta(k)} \cdot g_p^{-1}] \quad \forall 0 \leq k \leq n \quad (13)$$

is a Σ' -decomposition of S , for

$$\zeta(k) := \begin{cases} p + k & \text{if } k \leq n - p \\ k - (n - p + 1) & \text{if } k > n - p. \end{cases}$$

In particular, for $n = p = 1$, the previous point shows that $[g'_1] = [g_0 \cdot g_1^{-1}] \stackrel{(12)}{=} [g_1^{-1}] = [g_1]$ holds. ‡

From this, we easily conclude that

Lemma 3.6

In the situation of Definition 3.4, we have

$$g \cdot \iota(\Sigma_0) = \iota(\Sigma') \implies [g] = [g_k] \text{ and } \Sigma' = \Sigma_k \text{ for } 0 \leq k \leq n \text{ unique,} \quad (14)$$

for each segment $\Sigma' \subseteq S$, and each $g \in G$. In particular, for $n = 1$, there exists no other Σ -decomposition of S , and for $n \geq 2$, the only other Σ -decomposition is given by

$$\bar{\Sigma}_k := \Sigma_{\zeta(k)} \quad \text{and} \quad [\bar{g}_k] := [g_{\zeta(k)}] \quad \forall 1 \leq k \leq n, \quad (15)$$

for $\zeta \in S_n$ defined by $\zeta(k) = n - (k - 1)$ for $k = 1, \dots, n$.

PROOF: The uniqueness statement in (14) is clear, because the classes $[g_k]$ are mutually different. Moreover, since $S = \bigcup_{k=0}^n \Sigma_k$ holds, we have

$$\begin{aligned} g \cdot \iota(\Sigma_0) = \iota(\Sigma') &\implies g \cdot \iota|_{\Sigma_0} \sim_{\circ} \iota|_{\Sigma_k} \text{ for some } 0 \leq k \leq n, \text{ eqn} &\implies (g_k^{-1} \cdot g) \cdot \iota|_{\Sigma_0} \sim_{\circ} \iota|_{\Sigma_0} \\ &\implies [g] = [g_k] &\implies \Sigma' = \Sigma_k. \end{aligned}$$

Now, if $\Sigma'_0, \dots, \Sigma'_{n'}$ and $[g'_0], \dots, [g'_{n'}]$, define another Σ -decomposition of S , we necessarily have $\Sigma'_0 = \Sigma_0$ and $[g'_0] = [g_0]$. Moreover, (14) implies that $n' = n$ holds, and that there exists some unique $\zeta \in S_n$ with

$$(\Sigma'_k, [g'_k]) = (\Sigma_{\zeta(k)}, [g_{\zeta(k)}]) \quad \forall 1 \leq k \leq n.$$

Thus, the statement is clear for $n = 1$, and for $n \geq 2$, we conclude from the second point in Definition 3.4 that either $\Sigma'_1 = \Sigma_1$ or $\Sigma'_1 = \Sigma_n$ holds. Then, the same point shows that

- ▷ In the first case, we must have $\Sigma'_2 = \Sigma_2$, because $\Sigma'_2 = \Sigma_0 = \Sigma'_0$ cannot hold, and inductively we see that $\zeta(k) = k$ holds for all $k = 1, \dots, n$.
- ▷ In the second case, we must have $\Sigma'_2 = \Sigma_{n-1}$, because $\Sigma'_2 = \Sigma_0 = \Sigma'_0$ cannot hold, and inductively we see that $\zeta(k) = n - (k - 1)$ holds for all $k = 1, \dots, n$. ■

Finally, to show existence of such decompositions, let $\Sigma_0 \subset S$ be maximal with boundary points z_0, z_1 . We apply Proposition 3.1 and b) to $z := z_1$ and $\Sigma := \Sigma_0$, in order to find $[h_1] \neq [e]$ and $\Sigma_1 \subseteq S$ maximal with $z_1 \in \Sigma_0 \cap \Sigma_1$ and $h_1 \cdot \iota(\Sigma_0) = \Sigma_1$, whereby $\Sigma_0 \cap \Sigma_1$ only contains boundary points of Σ_0 .

Let $z_2 \neq z_1$ denote the other boundary point of Σ_1 , and let $\kappa: U(1) \rightarrow S$ be some homeomorphism with $\kappa(e^{i\alpha_i}) = z_i$ for $i = 0, 1$, for angles $0 = \alpha_0 < \alpha_1 \leq 2\pi$. Moreover, let $\alpha_2 \in (0, 2\pi]$ denote the unique angle, for which $\kappa(e^{i\alpha_2}) = z_2$ holds. Then, we must have $\alpha_1 < \alpha_2$, since otherwise $h_1 \cdot \iota|_{\Sigma_0} \sim_{\circ} \iota|_{\Sigma_0}$, hence $[h_1] = [e]$ holds. Now,

- If $\alpha_2 = 2\pi$ holds, we have $z_2 = z_0$, hence $S = \Sigma_0 \cup \Sigma_1$, $\Sigma_0 \cap \Sigma_1 = \{z_0, z_1\}$, and $h_1 \cdot \iota(\Sigma_0) = \iota(\Sigma_1)$.
- If $\alpha_2 < 2\pi$ holds, then for each $n \geq 2$, we can apply the above arguments inductively, in order to obtain maximal segments Σ_k with boundary points z_k, z_{k+1} , as well as classes $[h_k] \neq [e]$, with

$$h_k \cdot \iota(\Sigma_{k-1}) = \iota(\Sigma_k) \quad \forall 1 \leq k \leq n.$$

We define $g_k := h_k \cdot \dots \cdot h_1$ for $1 \leq k \leq n$, and let $0 = \alpha_0 < \dots, \alpha_{n+1} \leq 2\pi$ have the same meaning as above.

Then, if $\alpha_0 < \dots < \alpha_{n+1} < 2\pi$ holds for each such $n \geq 2$, then $2\pi \geq \alpha = \lim_n \alpha_n$ exists. Thus, for each neighbourhood V of $s := e^{i\alpha}$ in $U(1)$, we find some $n \in \mathbb{N}$ with $\kappa^{-1}(\Sigma_n) = e^{(i[\alpha_n, \alpha_{n+1}])} \subseteq V$. But, then for each neighbourhood U of $(\iota \circ \kappa)(s)$, we find some $n \in \mathbb{N}$ with $g_n \cdot \iota(\Sigma_0) \subseteq U$, which contradicts Property ii) of φ .

Thus, we find $n \geq 2$ with $\alpha_0 < \dots < \alpha_n < 2\pi$, such that either $\alpha_{n+1} = 2\pi$ or $0 < \alpha_{n+1} \leq \alpha_n$ holds. But, in the second case, we have

$$\begin{aligned} \iota|_{\Sigma_n} \sim_{\circ} \iota|_{\Sigma_0} &\implies g_n \cdot \iota|_{\Sigma_0} \sim_{\circ} \iota|_{\Sigma_0} &\implies [g_n] = [e], \\ &\implies \iota(\Sigma_n) = g_n \cdot \iota(\Sigma_0) = \iota(\Sigma_0) &\implies \alpha_n = 2\pi, \end{aligned}$$

which contradicts the choice of n . Thus, $0 = \alpha_0 < \dots < \alpha_{n+1} = 2\pi$, hence $z_{n+1} = z_0$ holds, with what $\Sigma_p \cap \Sigma_q$ is singleton for $|p - q| \in \{1, n\}$, and empty otherwise. Then, by construction, $g_k \cdot \iota(\Sigma_0) = \iota(\Sigma_k)$ holds for each $0 \leq k \leq n$ for $g_0 := e$.

Thus, we have shown the first part of

Lemma 3.7

Suppose that S is compact without boundary, and not a free segment. Then, each maximal segment $\Sigma \subseteq S$ is compact, and admits a Σ -decomposition of S that is unique in the sense of Lemma 3.6. The length of each such decomposition is the same for each maximal segment in S .

PROOF: The first statement is clear. For the second one, let Σ be maximal with Σ -decomposition $\Sigma_0, \dots, \Sigma_n$ and $[g_0], \dots, [g_n]$, and let Σ' be maximal with Σ' -decomposition $\Sigma'_0, \dots, \Sigma'_{n'}$ and $[g'_0], \dots, [g'_{n'}]$.

Then, if $\Sigma_p = \Sigma'_q$ holds for some $0 \leq p \leq n$ and $0 \leq q \leq n'$, it is immediate from (14) that each Σ_p equals some Σ'_q and vice versa, from which $n = n'$ is clear.

In the other case, the segments $\Sigma_0, \dots, \Sigma_n, \Sigma'_0, \dots, \Sigma'_{n'}$ are mutually different, and cannot be contained into each other by maximality. Thus, for each $0 \leq p \leq n$, $\text{int}[\Sigma_p]$ contains some boundary point of some Σ'_q , which then cannot be contained in Σ_k for each $k \neq p$. This shows $n \leq n'$, and in the same way, we conclude that $n' \leq n$ holds. ■

In the situation of Lemma 3.7, we define the **length** of S to be the length of one (and then each) decomposition of S .

3.2.2 Homeomorphic to an interval

If S is homeomorphic to an interval, we can argue as in the previous case, provided that Σ is compact maximal and contained in $\text{int}[S]$. For this, let \mathfrak{N} denote the set of all subsets of \mathbb{Z} , which are of the form⁸ $\mathfrak{n} = \{n \in \mathbb{Z} \mid \mathfrak{n}_- \leq n \leq \mathfrak{n}_+\}$ for $\mathfrak{n}_-, \mathfrak{n}_+ \in \mathbb{Z}_{\neq 0} \sqcup \{-\infty, \infty\}$ with $\mathfrak{n}_- < 0 < \mathfrak{n}_+$. Then,

Definition 3.8 (Σ -decomposition)

Let S be homeomorphic to an interval, and $\Sigma \subseteq \text{int}[S]$ a compact maximal segment. Then, by a Σ -decomposition of S , we understand a pair $(\{\Sigma_n\}_{n \in \mathfrak{n}}, \{[g_n]\}_{n \in \mathfrak{n}})$ consisting of classes $[g_n] \in \mathbb{G}$, as well as segments Σ_n on which ι is an embedding, such that $\Sigma_0 = \Sigma$ holds, and

- $\Sigma_p \cap \Sigma_q$ is singleton for $|p - q| = 1$, and empty otherwise.
- $g_n \cdot \iota(\Sigma_0) = \iota(\Sigma_n)$ holds for all $\mathfrak{n}_- < n < \mathfrak{n}_+$, as well as

$$g_{\mathfrak{n}_-} \cdot \iota(\Sigma_-) = \iota(\Sigma_{\mathfrak{n}_-}) \quad \text{iff} \quad \mathfrak{n}_- \neq -\infty \quad \text{and} \quad g_{\mathfrak{n}_+} \cdot \iota(\Sigma_+) = \iota(\Sigma_{\mathfrak{n}_+}) \quad \text{iff} \quad \mathfrak{n}_+ \neq \infty.$$

Here, $\Sigma_{\mathfrak{n}_\pm}$ denote boundary segments of S , and the Σ_\pm either equal, or are boundary segments of, Σ_0 .

Then, $[g_0] = [e]$ holds, because Σ_0 is free; and $[g_m] = [g_n]$ for $m \neq n$ implies $m, n \in \{\mathfrak{n}_-, \mathfrak{n}_+\}$ with $-\infty < \mathfrak{n}_- < \mathfrak{n}_+ < \infty$. Moreover, each Σ_n is free by Lemma 2.3 and Lemma 2.12, because $\iota|_{\Sigma_n}$ was required to be an embedding; and Σ_n is compact maximal if $g_n \cdot \iota(\Sigma_0) = \iota(\Sigma_n)$ holds.

Then,

Lemma and Remark 3.9

In the situation of Definition 3.8,

- 1) Let $\bar{\mathfrak{n}} := \{n \in \mathbb{Z} \mid \bar{\mathfrak{n}}_- \leq n \leq \bar{\mathfrak{n}}_+\} \in \mathfrak{N}$ for $\bar{\mathfrak{n}}_\pm := -\mathfrak{n}_\mp$. Then,

$$(\{\bar{\Sigma}_n\}_{n \in \bar{\mathfrak{n}}}, \{[\bar{g}_n]\}_{n \in \bar{\mathfrak{n}}}) \quad \text{with} \quad \bar{\Sigma}_n := \Sigma_{-n} \quad \text{and} \quad [\bar{g}_n] := [g_{-n}]$$

for each $n \in \bar{\mathfrak{n}}$, is obviously a Σ -decomposition of S .

⁸Of course, if $\mathfrak{n}_- = -\infty$, then $\mathfrak{n}_- \leq n$ just means that $n \in \mathbb{Z}$ holds, and analogously for \mathfrak{n}_+ .

2) Define $\Sigma' := \Sigma_p$ for some $\mathbf{n}_- < p < \mathbf{n}_+$, and $\mathbf{n}' := \{n \in \mathbb{Z} \mid \mathbf{n}'_- \leq n \leq \mathbf{n}'_+\} \in \mathfrak{N}$ for $\mathbf{n}'_{\pm} := \mathbf{n}_{\pm} - p$. Then,

$$(\{\Sigma'_n\}_{n \in \mathbf{n}'}, \{[g'_n]\}_{n \in \mathbf{n}'}) \quad \text{with} \quad \Sigma'_n := \Sigma_{n+p} \quad \text{and} \quad [g'_n] := [g_{n+p} \cdot g_p^{-1}] \quad (16)$$

for each $n \in \mathbf{n}'$, is obviously a Σ' -decomposition of S .

3) We have $S = \bigcup_{n \in \mathbf{n}} \Sigma_n$:

In fact, for $n \in \mathbf{n} - \{0\}$, let z_n denote the boundary point shared by Σ_n with

$$\Sigma_{n+1} \quad \text{for} \quad n \leq -1 \quad \quad \Sigma_{n-1} \quad \text{for} \quad 1 \leq n.$$

Moreover, let $\kappa: D \rightarrow S$ be some homeomorphism with $\kappa^{-1}(z_{-1}) < \kappa^{-1}(z_1)$, and define $\{a_n\}_{n \in \mathbf{n} - \{0\}}$ by

$$a_n := \kappa^{-1}(z_n) \quad \forall n \in \mathbf{n} - \{0\}.$$

Then, we have $a_m < a_n$ if $m < n$ holds for $m, n \in \mathbf{n}$, as well as $\kappa(A_n) = \Sigma_n$ for

$$A_n := [a_{n-1}, a_n] \quad \text{for} \quad \mathbf{n}_- < n \leq -1 \quad \quad A_0 := [a_{-1}, a_1] \quad \quad A_n := [a_n, a_{n+1}] \quad \text{for} \quad 1 \leq n < \mathbf{n}_+,$$

as well as

$$A_{\mathbf{n}_-} := D \cap (-\infty, a_{\mathbf{n}_-}] \quad \text{if} \quad \mathbf{n}_- > -\infty \quad \quad \text{and} \quad \quad A_{\mathbf{n}_+} := D \cap [a_{\mathbf{n}_+}, \infty) \quad \text{if} \quad \mathbf{n}_+ < \infty.$$

Then, in order to show the claim, it suffices to prove that $D = D' := \bigcup_{n \in \mathbf{n}} A_n$ holds. For this, assume that $t > a_1$ is not contained in D' .⁹ Then, we must have $\mathbf{n}_+ = \infty$, hence $\lim_{n \rightarrow \infty} a_n = a \leq t$ for some $a \in D$. Thus, for each $\epsilon > 0$, we find $n_{\epsilon} \geq 1$ with $A_{n_{\epsilon}} \subseteq [a - \epsilon, a]$, so that for each neighbourhood U of $(\iota \circ \kappa)(a)$, we find $n \geq 1$ with $g_n \cdot \iota(\Sigma_0) \subseteq U$. This, however, contradicts the Property ii) of φ . \ddagger

From this, we easily conclude that

Lemma 3.10

Suppose that we are in the situation of Definition 3.8, and that $\mathcal{O}, \mathcal{O}' \subseteq \text{int}[S]$ are open and connected, such that $\iota|_{\mathcal{O}}, \iota|_{\mathcal{O}'}$ are embeddings. Then, for $g \in G$, we have

$$(g \cdot \iota|_{\Sigma_0})(\mathcal{O}) = \iota(\mathcal{O}') \quad \implies \quad [g] = [g_n] \quad \text{and} \quad \mathcal{O}' \subseteq \Sigma_n \quad \text{for} \quad n \in \mathbf{n} \quad \text{unique}. \quad (17)$$

In particular, a Σ -decomposition is unique up to a change of orientation as in Remark 3.9.1.

PROOF: Since $S = \bigcup_{n \in \mathbf{n}} \Sigma_n$ holds by Lemma and Remark 3.9.3, we have

$$\begin{aligned} (g \cdot \iota|_{\Sigma_0})(\mathcal{O}) = \iota(\mathcal{O}') &\implies g \cdot \iota|_{\Sigma_0} \sim_{\circ} \iota|_{\Sigma_n} \quad \text{for some } n \in \mathbf{n} \\ &\implies (g_n^{-1} \cdot g) \cdot \iota|_{\Sigma_0} \sim_{\circ} \iota|_{\Sigma_0} \implies [g] = [g_n]. \end{aligned}$$

Here, n is unique if $n \neq \mathbf{n}_{\pm}$ holds by the last statement in Definition 3.8; and then $\mathcal{O}' \subseteq \Sigma_n$ is clear from $g_n \cdot \iota(\Sigma_0) = \iota(\Sigma_n)$, because both $g_n \cdot \iota$ and ι are injective. Thus, if $n \in \{\mathbf{n}_-, \mathbf{n}_+\}$ holds, we must have $\mathcal{O}' \cap (\bigcup_{n \in \mathbf{n} - \{\mathbf{n}_-, \mathbf{n}_+\}} \Sigma_n) = \emptyset$, so that \mathcal{O}' is either contained in $\Sigma_{\mathbf{n}_-}$ (for $\mathbf{n}_- \neq -\infty$) or in $\Sigma_{\mathbf{n}_+}$ (for $\mathbf{n}_+ \neq \infty$), as it is connected. This shows (17), from which the last statement follows by the same elementary arguments, we have used in the second part of Lemma 3.6. \blacksquare

Now, to construct such a decomposition for $\Sigma = \Sigma_0 \subseteq \text{int}[S]$ compact and maximal, let z_{-1} and z_1 denote the boundary points of Σ . We apply Proposition 3.1 to Σ and $z := z_1$, in order to fix $[h_1] \neq [e]$, a compact boundary segment Σ_b of Σ , and a compact free segment $\Sigma_z \subseteq S$, such that

$$\Sigma_0 \cap \Sigma_z = \{z\} \quad \quad h_1 \cdot \iota(\Sigma_b) = \iota(\Sigma_z) \quad \quad \text{and} \quad \quad h_1 \cdot \iota(b) = \iota(z)$$

holds. In particular, then $h_1 \cdot \iota|_{\mathcal{O}} = \iota \circ \rho$ holds for some analytic diffeomorphism $\rho: \Sigma_b \supseteq \mathcal{O} \rightarrow \mathcal{O}' \subseteq S$ with \mathcal{O} open and connected, for which $\bar{\rho}$ must necessarily be defined on Σ_b by Lemma 2.3. Thus, by a), exactly one of the following situations holds.

⁹The case where $t < a_{-1}$ is not contained in D' follows analogously.

- We have $h_1 \cdot \iota(\Sigma_0) = \iota(\Sigma_1)$ with $\{z_1\} = \Sigma_0 \cap \Sigma_1$ for a compact segment $\Sigma_1 \subseteq S$, and define $\Sigma_+ := \Sigma_0$ if Σ_1 is a boundary segment of S .
- We have $h_1 \cdot \iota(\Sigma_+) = \iota(\Sigma_1)$ with $\{z_1\} = \Sigma_0 \cap \Sigma_1$ for a boundary segment Σ_+ of Σ_0 , and a boundary segment Σ_1 of S on which ι is an embedding.

If Σ_1 is a boundary segment of S , we define $n_+ := 1$, and have done. In the other case, Σ_1 is contained in $\text{int}[S]$, so that we can apply the above arguments to Σ_1 and its boundary point $z_2 \neq z_1$. Similarly, we can apply these arguments to Σ_0 and z_{-1} , so that we inductively obtain a Σ -decomposition of S . Thus,

Lemma 3.11

If S is homeomorphic to an interval, each compact maximal segment $\Sigma \subseteq \text{int}[S]$ admits a Σ -decomposition of S that is unique in the sense of Lemma 3.10.

3.3 z -Decompositions

So far, we have discussed the situation where S admits a compact maximal segment which is properly contained in $\text{int}[S]$. This is always the case if S is compact without boundary and not a free segment by itself; and we now are going to show that exactly one other case can occur if S is homeomorphic to an interval. More precisely, we will show that

Proposition 3.12

If S is homeomorphic to an interval and not a free segment, it either admits a unique z -decomposition or a compact maximal segment contained in $\text{int}[S]$.

Here, by a z -decomposition, we understand the following.

Definition 3.13 (z -decomposition)

Let S be homeomorphic to an interval, and $z \in \text{int}[S]$. Then, a z -decomposition of S is a class $[g] \neq [e]$ with $g \in G_z$, such that $g \cdot \iota(\mathcal{K}_-) = \iota(\mathcal{K}_+)$ holds for some compact segments $\mathcal{K}_\pm \subseteq S$ with $\mathcal{K}_+ \cap \mathcal{K}_- = \{z\}$. In addition to that, we require that the unique boundary segments Σ_\pm of S , for which $\Sigma_+ \cap \Sigma_- = \{z\}$ and $\mathcal{K}_\pm \subseteq \Sigma_\pm$ holds, are free.

Then, a) applied to the analytic diffeomorphism $\rho := \iota^{-1} \circ (g \cdot \iota|_{\mathcal{K}_-}) : \mathcal{K}_- \rightarrow \mathcal{K}_+$ and $\Sigma = \Sigma_-$, shows that exactly one of the following two situations holds:

$$\iota^{-1} \circ (g \cdot \iota|_{C_-}) : \Sigma_- \supset C_- \rightarrow \Sigma_+ \quad \text{is an analytic diffeomorphism,} \quad (18)$$

$$\iota^{-1} \circ (g \cdot \iota|_{\Sigma_-}) : \Sigma_- \rightarrow C_+ \subseteq \Sigma_+ \quad \text{is an analytic diffeomorphism,} \quad (19)$$

for $C_\pm \subseteq \Sigma_\pm$ segments with $\mathcal{K}_\pm \subseteq C_\pm$.

Now, in the situation of the above definition, we can interchange the roles of \mathcal{K}_- and \mathcal{K}_+ , in order to obtain the z -decomposition $[g^{-1}] \neq [e]$ of S . However, then the same arguments we have used for (9) show that $g^{-1} \cdot \iota|_{\Sigma_-} \sim_\circ \iota$ holds, hence $[g^{-1}] = [g]$ by the next lemma. Moreover, the first part of this lemma shows that iff a z -decomposition exists, it is even unique in the sense that there exists no z' -decomposition for any other $z \neq z' \in \text{int}[S]$.

Lemma 3.14

In the situation of Definition 3.13, the segments Σ_\pm are the only maximal ones, and we have

$$g' \cdot \iota|_{\Sigma_-} \sim_\circ \iota \quad \text{for } g' \in G \quad \implies \quad [g'] \in \{[e], [g]\}. \quad (20)$$

PROOF: If $C \subseteq S$ is a free segment, z cannot be contained in the interior of C , because then $g \cdot \iota|_C \sim_\circ \iota|_C$, hence $[g] = [e]$ would hold. Thus, we either have $C \subseteq \Sigma_-$ or $C \subseteq \Sigma_+$, so that the first statement is clear as Σ_\pm are free segments by definition.

Now, the left hand side of (20) implies that $g' \cdot \iota|_{\Sigma_-} \sim_\circ \iota|_{\Sigma_-}$ or $g' \cdot \iota|_{\Sigma_-} \sim_\circ \iota|_{\Sigma_+}$ holds. In the first case, we have $[g'] = [e]$, because Σ_- is a free segment. In the second case, since either (18) or (19) holds, we have

$$(18): \quad g' \cdot \iota|_{\Sigma_-} \sim_\circ g \cdot \iota|_{\Sigma_-} \implies [g'] = [g],$$

$$(19): \quad g'^{-1} \cdot \iota|_{\Sigma_+} \sim_\circ g^{-1} \cdot \iota|_{\Sigma_+} \xrightarrow{(5)} [g'] = [g],$$

because the left hand side of the second line implies $g' = h \cdot g = g \cdot (g^{-1} \cdot h \cdot g)$ for some $h \in G_S$. ■

Corollary 3.15

Suppose that S is homeomorphic to an interval. Then, if S admits a z -decomposition for some $z \in \text{int}[S]$, it admits no other decomposition. Conversely, if S admits a Σ -decomposition, it cannot admit a z -decomposition for some $z \in \text{int}[S]$.

PROOF: If S admits a z -decomposition, it cannot admit a compact maximal segment contained in $\text{int}[S]$ and vice versa, just by the first part of Lemma 3.14. The rest already has been clarified above. ■

We now are ready for the

PROOF (OF PROPOSITION 3.12): By Corollary 3.15, it suffices to show that S admits a z -decomposition for some $z \in \text{int}[S]$ if it admits no compact maximal segment contained in $\text{int}[S]$. For this, assume that the latter statement holds, and let $\Sigma_- \subset S$ be a maximal segment. Then, Σ_- must be a boundary segment of S as it is closed, and we denote the unique boundary point of Σ_- that is contained in $\text{int}[S]$ by z .

Let Σ_+ denote the unique boundary segment of S , for which $\Sigma_- \cap \Sigma_+ = \{z\}$ holds. We now first show that Σ_+ is a free segment. For this, let us apply the statement following Lemma 2.13, in order to conclude that there exists a free segment $\Sigma_z \subseteq \Sigma_+$ containing z . Then, by Lemma 2.11, we find a maximal free segment $\Sigma' \subseteq S$ with $\Sigma_z \subseteq \Sigma'$, which must be a boundary segment of S as well. Since Σ_- is maximal with $\Sigma_z - \Sigma_- \neq \emptyset$, it cannot be contained in Σ' . Thus, we must have $\Sigma_+ \subseteq \Sigma'$, so that Σ_+ is a free segment.

Then, in order to prove the claim, it suffices to show that there exists a compact segment $\Sigma' \subseteq \text{int}[S]$ with $z \in \text{int}[\Sigma']$, such that $\Sigma := \Sigma_- \cap \Sigma'$ is compact and maximal w.r.t. $S' := \Sigma'$. In fact, then Lemma 3.3 shows that Σ is negative w.r.t. S' ; so that we find $[g] \neq [e]$ with $g \in G_z$, such that $g \cdot \iota(\mathcal{K}_-) = \iota(\mathcal{K}_+)$ holds for compact boundary segments \mathcal{K}_\pm of Σ_\pm containing z .

Thus, assume that such a compact segment Σ' does not exist, and let $\kappa: D \rightarrow S$ be a homeomorphism with $\kappa(D \cap (-\infty, x]) = \Sigma_-$ for $x := \kappa^{-1}(z)$. We fix $x < t < \sup[D]$, and define $\Sigma'_n := \kappa([a_n, t])$ for each $n \in \mathbb{N}$, for some sequence $\{a_n\}_{n \in \mathbb{N}} \subseteq D$ with

$$\inf[D] < a_n < x < t < \sup[D] \quad \text{and} \quad a_{n+1} < a_n \quad \forall n \in \mathbb{N},$$

as well as $\lim_n a_n = \inf[D]$. By assumption, $\Sigma_- \cap \Sigma'_n = \kappa([a_n, x])$ is not maximal w.r.t. Σ'_n ; but it is a free segment, because Σ_- is a free segment. Thus, we find $x < t_n \leq t$, such that $\kappa([a_n, t_n])$ is maximal w.r.t. Σ'_n . If $t_n = t$ holds for all $n \in \mathbb{N}$, then $\kappa(D \cap (-\infty, t])$ is free and properly contains Σ_- , which contradicts maximality of Σ_- . Thus, $\kappa([a_n, t_n])$ is maximal w.r.t. Σ'_n for some $\inf[D] < a_n < x < t_n < t$, hence negative by Lemma 3.3. Thus, we find $[g] \neq [e]$ with $g \cdot \iota|_{\kappa([x, t_n])} \sim_\circ \iota|_{\kappa([t_n, t])}$, which contradicts that Σ_+ is a free segment. ■

3.4 Σ -Decompositions

In this final subsection, we will investigate the case where S admits a compact maximal segment $\Sigma \subset \text{int}[S]$ in more detail. In the first part, we will show that all such segments are either positive or negative, provided that S is fixed. Then, we will consider these two cases in more detail, in particular, providing explicit formulas for the classes $[g_n]$ occurring in a Σ -decomposition of S .

In the following, if we write $S \cong \text{U}(1)$ or $S \cong \text{D}$, we will mean that S is homeomorphic to $\text{U}(1)$ or to some interval, respectively.

3.4.1 Decomposition types

Let us start with the observation that if $\Sigma \subset \text{int}[S]$ is positive/negative, each compact maximal segment occurring in a Σ -decomposition is positive/negative as well.

In fact, if $\Sigma_-, \Sigma_+ \subseteq S$ are compact maximal segments with common boundary point z , then

$$g \cdot \iota(\Sigma_-) = \iota(\Sigma_+) \quad \Longleftrightarrow \quad g^{-1} \cdot \iota(\Sigma_+) = \iota(\Sigma_-),$$

whereby we have $g \in G_z$ iff $g^{-1} \in G_z$ holds. Thus, Σ_- is positive/negative iff Σ_+ is positive/negative, from which the statement follows inductively. ‡

Then, if S is not a free segment, and admits no z -decomposition, we will say that S is **positive/negative** iff it admits a positive/negative segment. This definition makes sense, because

Lemma 3.16

If $\Sigma \subset \text{int}[S]$ is negative, the segments occurring in a Σ -decomposition of S are maximal, and the only maximal ones. Moreover, each compact maximal segment occurring in this decomposition is negative.

In particular, if S is not a free segment and admits no z -decomposition, all compact maximal segments are either positive or negative.

PROOF: If Σ is negative, each compact maximal segment occurring in a Σ -decomposition of S is negative, just by what we have already shown above. Thus, if $C \subseteq S$ is a free segment, then $\text{int}[C]$ cannot contain any boundary point of any segment occurring in this Σ -decomposition, just by the same arguments as in Lemma 3.14. Consequently, C must be contained in one of these segment, from which the first statement is clear. The second statement is now obvious. ■

Then, Lemma 3.7, Lemma 3.11, Proposition 3.12 and Lemma 3.16 prove

Theorem 3.17

Let φ be non-contractive, and (S, ι) free but not a free segment by itself. Then, S either admits a unique z -decomposition or a compact maximal segment properly contained in $\text{int}[S]$; whereby the first case cannot occur if S is compact without boundary. In the second case, S is either positive or negative, and admits a Σ -decomposition for each maximal segment $\Sigma \subset \text{int}[S]$ that is unique in the sense of Lemma 3.6, and Lemma 3.10 in the respective cases.

Moreover, from the first part of Lemma 3.16, we easily obtain

Corollary 3.18

If S is compact without boundary and negative, then the length of S is odd.

PROOF: Let $\Sigma \subset S$ be negative with Σ -decomposition $\Sigma_0, \dots, \Sigma_n$ and $[g_0], \dots, [g_n]$ for $n \geq 2$. We write

$$\{z_0\} = \Sigma_0 \cap \Sigma_n \quad \text{as well as} \quad \{z_k\} = \Sigma_{k-1} \cap \Sigma_k \quad \text{for} \quad k = 1, \dots, n,$$

and define $h_k = g_k \cdot g_{k-1}^{-1}$ for $k = 1, \dots, n$. Then, we have

$$g_n \cdot \iota(\Sigma_0) = \iota(\Sigma_n) \quad \text{as well as} \quad h_k \cdot \iota(\Sigma_{k-1}) = \iota(\Sigma_k) \quad \text{for} \quad k = 1, \dots, n,$$

whereby $g_n \in G_{z_0}$ and $h_k \in G_{z_k}$ holds for $k = 1, \dots, n$, by negativity of each Σ_k . Now, assume that n is even. Then, since

$$h_{k+2} \cdot h_{k+1} \cdot \iota(z_k) = h_{k+2} \cdot \iota(z_{k+2}) = \iota(z_{k+2})$$

holds for $k = 0, \dots, n-2$, we have

$$\iota(z_0) = g_n \cdot \iota(z_0) = [h_n \cdot h_{n-1}] \cdot \dots \cdot [h_2 \cdot h_1] \cdot \iota(z_0) = \iota(z_n),$$

which contradicts the definitions. ■

3.4.2 Positive decompositions

We will now investigate the case where S is positive in more detail. For this, we first observe that

Lemma 3.19

If $S \cong D$ is positive with Σ -decomposition $(\{\Sigma_n\}_{n \in \mathbb{N}}, \{[g_n]\}_{n \in \mathbb{N}})$, then we have

$$[g_n] = [g^n] \quad \forall n \in \mathbb{N} \quad \text{for each} \quad g \in [g_1]. \quad (21)$$

If $S \cong U(1)$ is positive with Σ -decomposition $\Sigma_0, \dots, \Sigma_n$ and $[g_0], \dots, [g_n]$, then we have

$$[g_k] = [g^k] \quad \text{for} \quad k = 0, \dots, n \quad \text{for each} \quad g \in [g_1], \quad (22)$$

whereby $[g^n] = [g^{-1}]$ holds by (10).

PROOF: Since (22) follows in the same way, only show (21), for which we can assume that $g = g_1$ holds. We write $\{z_k\} = \Sigma_{k-1} \cap \Sigma_k$ for $1 \leq k \in \mathbf{n}$, and observe that

$$g_n = h_n \cdot \dots \cdot h_1 \quad \forall 1 \leq n \in \mathbf{n} \quad \text{holds for} \quad h_k = g_k \cdot g_{k-1}^{-1} \quad \forall 1 \leq k \in \mathbf{n}, \quad (23)$$

whereby we have $h_k \cdot \iota(\Sigma_{k-1}) = \iota(\Sigma_k)$ for each $1 \leq k \in \mathbf{n}$. Thus, $[h_k]$ is the unique class from Proposition 3.1, applied to Σ_{k-1} and z_k ; and $[(h_{k-1})^{-1}]$ is the unique class from Proposition 3.1 applied to Σ_{k-1} and z_{k-1} for $k \geq 2$. Then, since Σ_{k-1} is positive, we have

$$[h_k] \stackrel{(10)}{=} [((h_{k-1})^{-1})^{-1}] = [h_{k-1}] \quad \forall 2 \leq k \in \mathbf{n},$$

hence $[h_k] = [g]$ for all $1 \leq k \in \mathbf{n}$, because $h_1 = g_1 = g$ holds by definition. Thus, (21) is clear from (23) and (12) for $1 \leq n \in \mathbf{n}$; and since $[g_{-1}] = [g_1^{-1}] = [g^{-1}]$ holds by (10), we can argue in the same way, in order to show that (21) also holds for all $\mathbf{n} \ni n \leq -1$. \blacksquare

For the rest of this subsection, let us assume that we are in the situation of Lemma 3.19 with fixed Σ -decomposition \mathcal{S} of S . We now want to figure out, which classes can occur on the right hand side of (21) and (22) for any further positive segment $\Sigma' \subset \text{int}[S]$.

First, if Σ' equals some Σ_p occurring in \mathcal{S} , then for the Σ' -decomposition¹⁰

- from Lemma and Remark 3.9.2 if $S \cong D$ holds
- from Remark 3.5.4 if $S \cong U(1)$ holds,

we necessarily have $[g'_1] = [g]$, hence

- $[g'_n] = [g^n]$ for all $n \in \mathbf{n}'$ if $S \cong D$ holds,
- $[g'_k] = [g^k]$ for $k = 0, \dots, n$ if $S \cong U(1)$ holds.

This is clear from (21), (22), the definitions of the classes $[g'_q]$, and (12). We will call the Σ' -decomposition of S singled out in this way, **\mathcal{S} -oriented** in the following.

Second, if $\Sigma' \neq \Sigma_p$ holds for each Σ_p occurring in \mathcal{S} , we find some

- $\mathbf{n}_- < p < \mathbf{n}_+$ if $S \cong D$ holds
- $0 \leq p \leq n$ if $S \cong U(1)$ holds,

such that either

- I) $\Sigma' \subseteq \text{int}[\Sigma_{p-1} \cup \Sigma_p]$ and $\Sigma_{p-1} \cap \Sigma_p \ni z \in \text{int}[\Sigma']$ or
- II) $\Sigma' \subseteq \text{int}[\Sigma_p \cup \Sigma_{p+1}]$ and $\Sigma_p \cap \Sigma_{p+1} \ni z \in \text{int}[\Sigma']$ holds,

for z a (necessarily unique) common boundary point of Σ_{p-1} and Σ_p or Σ_{p+1} and Σ_p , respectively. Here (and in the following), we define $\Sigma_{-1} := \Sigma_n$ and $\Sigma_{n+1} := \Sigma_0$ for the case that $S \cong U(1)$ holds.

In fact, the above statement is clear from maximality of Σ' , as well as maximality of the segments Σ_q for $\mathbf{n}_- < q < \mathbf{n}_+$ if $S \cong D$, as well as $0 \leq q \leq n$ if $S \cong U(1)$ holds. For this observe that Σ' cannot be contained in $\Sigma_{\mathbf{n}_-}$ (for $-\infty < \mathbf{n}_-$) or in $\Sigma_{\mathbf{n}_+}$ (for $\mathbf{n}_+ < -\infty$) if $S \cong D$ holds, because these segments are free, and since $\Sigma' \subset \text{int}[S]$ is compact and maximal. Now,

- I) In the first case, we let z_- and z_+ denote the boundary points of Σ' , which are contained in $\text{int}[\Sigma_{p-1}]$ and $\text{int}[\Sigma_p]$, respectively. Moreover, we consider the unique Σ' -decomposition of S with $\{z_{\pm}\} = \Sigma'_0 \cap \Sigma'_{\pm 1}$, and call it **\mathcal{S} -oriented** as well.¹¹

¹⁰Recall that, by Lemma and Remark 3.9.1 for $S \cong D$, as well as Lemma 3.6 for $S \cong U(1)$, there exists only one other Σ' -decomposition of S .

¹¹Again, there exists only one further Σ' -decomposition of S .

Since Σ' is positive, $g'_1 \cdot \iota(\Sigma'_{z_-}) = \iota(\Sigma'_{z_+})$ holds for a compact boundary segment Σ'_{z_-} of Σ' containing z_- , and a compact boundary segment Σ'_{z_+} of Σ'_1 containing z_+ . Then, since $z_- \in \text{int}[\Sigma_{p-1}]$ and $z_+ \in \text{int}[\Sigma_p]$ holds, we have

$$g'_1 \cdot \iota|_{\Sigma_{p-1}} \sim_\circ \iota|_{\Sigma_p} \implies [g'_1] = [g]. \quad (24)$$

For this observe that for $S \cong \text{U}(1)$ and $p = 0$, the left hand side of (24) shows $(g'_1)^{-1} \cdot \iota|_{\Sigma_0} \sim_\circ \iota|_{\Sigma_n}$; so that in this case, we have

$$[(g'_1)^{-1}] = [g^n] = [g^{-1}] \implies g'_1 = h \cdot g \text{ for some } h \in G_S,$$

hence $[g'_1] = [g]$ by Corollary 2.8. Here, the second equality on the left hand side is due to Lemma 3.19.

II) In the second case, we let z_- and z_+ denote the boundary points of Σ' , which are contained in $\text{int}[\Sigma_p]$ and $\text{int}[\Sigma_{p+1}]$, respectively, and consider the unique Σ' -decomposition of S (**\mathcal{S} -oriented**), for which $\{z_\pm\} = \Sigma'_0 \cap \Sigma'_{\pm 1}$ holds. Then, the same arguments as above show

$$g'_1 \cdot \iota|_{\Sigma_p} \sim_\circ \iota|_{\Sigma_{p+1}} \implies [g'_1] = [g],$$

as for $S \cong \text{U}(1)$ and $p = n$, the left hand side reads $(g'_1)^{-1} \cdot \iota|_{\Sigma_0} \sim_\circ \iota|_{\Sigma_n}$ as well.

Thus, we have shown that

Lemma 3.20

In the situation of Lemma 3.19, the class occurring on the right hand sides of (21) and (22), is the same for each \mathcal{S} -oriented Σ' -decomposition of S .

Finally, let us clarify that, in the positive case, S indeed admits much more positive segments than only those occurring in \mathcal{S} . For this, let Σ' be a compact segment with boundary points z_\pm , such that $g \cdot \iota(z_-) = \iota(z_+)$ and $\Sigma' \subseteq \Sigma_p \cup \Sigma_{p+1}$ holds for some

- $\mathbf{n}_- \leq p < \mathbf{n}_+$ if $S \cong \text{D}$
- $0 \leq p \leq n$ if $S \cong \text{U}(1)$,

whereby $\text{int}[\Sigma']$ contains exactly one common boundary point z of Σ_p and Σ_{p+1} . Then, Σ' is the union of the free compact segments $\Sigma_- := \Sigma' \cap \Sigma_p$ and $\Sigma_+ := \Sigma' \cap \Sigma_{p+1}$, only sharing the point z .¹²

We now show that Σ' is maximal, for which we first recall that

$$q \cdot \iota|_{\Sigma_p} \sim_\circ \iota|_{\Sigma_{p+1}} \text{ for } q \in G \implies [q] = [g]. \quad (25)$$

Now, it is immediate from positivity that we have

- $g \cdot \iota(\Sigma_-) \cap \iota(\Sigma_+) = \{z_+\}$, as $g \cdot \iota(z_-) = \iota(z_+)$ holds.
- $g \cdot \iota|_{\Sigma_-} \sim_\circ \iota|_{\Sigma_+}$ for each segment $\Sigma_{--} \subseteq \Sigma_p$ with $z \in \Sigma_- \subset \Sigma_{--}$.
- $g^{-1} \cdot \iota|_{\Sigma_{++}} \sim_\circ \iota|_{\Sigma_-}$ for each segment $\Sigma_{++} \subseteq \Sigma_{p+1}$ with $z \in \Sigma_+ \subset \Sigma_{++}$.

Thus, Σ' is a free segment, because for $[q] \neq [e]$, we have

$$\begin{aligned} q \cdot \iota|_{\Sigma'} \sim_\circ \iota|_{\Sigma'} &\implies q \cdot \iota|_{\Sigma_-} \sim_\circ \iota|_{\Sigma_+} \quad \text{or} \quad q^{-1} \cdot \iota|_{\Sigma_-} \sim_\circ \iota|_{\Sigma_+} \\ &\implies q \cdot \iota|_{\Sigma_p} \sim_\circ \iota|_{\Sigma_{p+1}} \quad \text{or} \quad q^{-1} \cdot \iota|_{\Sigma_p} \sim_\circ \iota|_{\Sigma_{p+1}} \\ &\implies g \cdot \iota|_{\Sigma_-} \sim_\circ \iota|_{\Sigma_+}, \end{aligned}$$

which contradicts the first point. Here, for the first implication, we have used that Σ_- and Σ_+ are free segments, and the third one is clear from (25), and the first line. Then, Σ' is even maximal by the last two points, because $[g], [g^{-1}] \neq [e]$ holds.

For this, observe that $\Sigma_- = \Sigma_p$ or $\Sigma_+ = \Sigma_{p+1}$ can only hold for $S \cong \text{D}$ and $-\infty < \mathbf{n}_- = p$ or $p+1 = \mathbf{n}_+ < \infty$, whereby then Σ_- or Σ_+ are boundary segments of S , respectively. In other words, if $\overline{\Sigma}$ is a free segment properly containing Σ' , we must have $\Sigma_- \subset \Sigma_{--} := \overline{\Sigma} \cap \Sigma'$ or $\Sigma_+ \subset \Sigma_{++} := \overline{\Sigma} \cap \Sigma'$, which contradicts the last two points.

Thus, we in particular have shown that

¹²This is clear if $S \cong \text{D}$ holds, and follows from maximality of Σ_p and Σ_{p+1} in the other case.

Corollary 3.21

If S is positive, then for each $z \in \text{int}[S]$, we find $\Sigma \subset S$ positive with $z \in \text{int}[\Sigma]$.

3.4.3 Negative decompositions

For the rest of this section, let $\Sigma \subset \text{int}[S]$ be negative with fixed Σ -decomposition \mathcal{S} . We define the map $\sigma: \mathbb{Z}_{\neq 0} \rightarrow \{-1, 1\}$ by

$$\sigma(n) := \begin{cases} (-1)^{n-1} & \text{if } n > 0 \\ (-1)^n & \text{if } n < 0, \end{cases}$$

and finally want to show the following formulas for the classes occurring in \mathcal{S} :

- If $S \cong D$ holds with $\mathcal{S} = (\{\Sigma_n\}_{n \in \mathbf{n}}, \{[g_n]\}_{n \in \mathbf{n}})$, we have

$$[g_n] = [g_{\sigma(\text{sign}(n))} \cdot \dots \cdot g_{\sigma(n)}] \quad \forall n \in \mathbf{n} - \{0\}. \quad (26)$$

- If $S \cong U(1)$ holds with \mathcal{S} given by $\Sigma_0, \dots, \Sigma_n$ and $[g_0], \dots, [g_n]$, we have

$$[g_k] = [g_{\sigma(1)} \cdot \dots \cdot g_{\sigma(k)}] \quad \forall 1 \leq k \leq n \quad \text{for} \quad g_{-1} := g_n. \quad (27)$$

Here, it suffices to show the first case, because in the second one, the statement is clear for $n = 1$. Moreover, for $n \geq 2$, we can choose an open connected segment $S' \subset S$ containing $\Sigma_0, \dots, \Sigma_{n-1}$, and let Σ'_{-1} and Σ'_n denote the components of $S' \cap \Sigma_n$, sharing a boundary point with Σ_0 and Σ_{n-1} , respectively. Then, we consider the Σ -decomposition $\mathcal{S}' = (\{\Sigma'_n\}_{n \in \mathbf{n}}, \{[g'_n]\}_{n \in \mathbf{n}})$ of S' , defined by $\mathbf{n} = \{-1, 0, 1, \dots, n\}$, as well as

- $[g'_k] := [g_k]$ for $k = 0, \dots, n$, and $[g_{-1}] := [g_n]$,
- $\Sigma'_k := \Sigma_k$ for $k = 0, \dots, n-1$.

Then, (26) holds for \mathcal{S} if (27) holds for \mathcal{S}' , so that we only have to show the case where $S \cong D$ holds. For this, let us define

$$h_n := g_n \cdot g_{n+1}^{-1} \quad \text{for } \mathbf{n}_- \leq n \leq -1 \quad \text{as well as} \quad h_n := g_n \cdot g_{n-1}^{-1} \quad \text{for } 1 \leq n \leq \mathbf{n}_+,$$

and observe that then by (9), we have

$$[h_n] = [h_n^{-1}] \quad \forall n \in \mathbf{n} \quad \text{hence} \quad [g_{\pm 1}] = [h_{\pm 1}] = [h_{\pm 1}^{-1}] = [g_{\pm 1}^{-1}]. \quad (28)$$

Step I

Now, let us first show that

$$\begin{aligned} \mathbf{n}_- \leq -2: \quad [h_{n-1}] &= [h_n \cdot h_{n+1} \cdot h_n] & \forall \mathbf{n}_- < n \leq -2 & \quad \text{and} \quad [h_{-2}] = [g_{-1} \cdot g_1 \cdot g_{-1}], \\ \mathbf{n}_+ \geq 2: \quad [h_{n+1}] &= [h_n \cdot h_{n-1} \cdot h_n] & \forall 2 \leq n < \mathbf{n}_+ & \quad \text{and} \quad [h_2] = [g_1 \cdot g_{-1} \cdot g_1]. \end{aligned} \quad (29)$$

For this, let $\Sigma_{\pm} \subseteq \text{int}[S]$ be negative with $\Sigma_- \cap \Sigma_+ = \{z\}$, and $h \cdot \iota(\Sigma_-) = \iota(\Sigma_+)$ for $[h] \neq [e]$. Moreover, let $\Sigma_{\pm\pm} \subseteq S$ be closed with $\Sigma_{\pm\pm} \cap \Sigma_{\pm} = \{z_{\pm}\}$ for $z_{\pm} \neq z$, as well as

$$h_- \cdot \iota(\Sigma_{z_-}) = \iota(\Sigma_{--}) \quad \text{and} \quad h_+ \cdot \iota(\Sigma_{z_+}) = \iota(\Sigma_{++}) \quad \text{for some} \quad h_{\pm} \in G_{\pm z} - G_S, \quad (30)$$

for boundary segments $\Sigma_{z_{\pm}}$ of Σ_{\pm} containing z_{\pm} . Then, (29) is clear, if we show that

$$[h_-] = [h \cdot h_+ \cdot h] \quad \text{as well as} \quad [h_+] = [h \cdot h_- \cdot h] \quad \text{holds.}$$

Now,

▷ Since we have $h \cdot \iota(\Sigma_-) = \iota(\Sigma_+)$ with $h \cdot \iota(z_-) = \iota(z_+)$ for $z_{\pm} \in \text{int}[S]$, Corollary 2.2 shows that

$$h \cdot \iota(\Sigma'_-) = \iota(\Sigma'_+) \quad (31)$$

holds for some compact boundary segments Σ'_{\pm} of $\Sigma_{\pm\pm}$ containing z_{\pm} .

▷ Then, combining (31) with (30), we conclude that

$$h_+ \cdot \iota|_{\Sigma_+} \sim_{\circ} (h \cdot h_- \cdot h) \cdot \iota|_{\Sigma_+} \implies [h_+] = [h \cdot h_- \cdot h] \quad \text{as } \Sigma_+ \text{ is free,}$$

whereby we have used that $h \cdot \iota(\Sigma_+) = \iota(\Sigma_-)$ holds by (28).

▷ Thus, we find $q \in G_S$ with

$$h_+ \cdot q = h \cdot h_- \cdot h \implies [h_-] = [h \cdot q' \cdot h_+ \cdot q \cdot h] \stackrel{(5)}{=} [h \cdot q' \cdot (h_+ \cdot h)] \stackrel{(5)}{=} [h \cdot h_+ \cdot h]$$

for $q' \in G_S$ with $h^{-1} = h \cdot q'$ by (9). For the last equality, we have used that $(h_+ \cdot h) \cdot \iota|_{\Sigma_-} \sim_{\circ} \iota|_{\Sigma_{++}}$, hence $(h_+ \cdot h) \in \mathcal{O}(S)$ holds.

Step II

Next, let us derive from (29) that

$$\begin{aligned} [h_n] &= [g_{-1} \cdot (g_1 \cdot g_{-1})^{|n|-1}] & \forall \mathbf{n}_- \leq n \leq -1 \\ [h_n] &= [g_1 \cdot (g_{-1} \cdot g_1)^{n-1}] & \forall 1 \leq n \leq \mathbf{n}_+ \end{aligned} \quad (32)$$

holds. For this, we first observe that $g_{\pm 1}^{-1} \in \mathcal{O}(S)$ implies that for each $q \in G_S$, we have

$$[g_{\pm 1} \cdot q \cdot g_{\pm 1}] \stackrel{(28)}{=} [g_{\pm 1} \cdot q \cdot g_{\pm 1}^{-1}] \stackrel{(5)}{=} [e] \implies q_n^{\pm} := (g_{\mp 1} \cdot g_{\pm 1})^n \cdot (g_{\pm 1} \cdot g_{\mp 1})^n \in G_S \quad \forall n \in \mathbb{N}, \quad (33)$$

which follows just inductively from the left hand side. In addition to that, we have

$$\mathbf{n}_- \leq -2 \implies g_1 \cdot g_{-1} \in \mathcal{O}(S) \quad \text{as well as} \quad \mathbf{n}_+ \geq 2 \implies g_{-1} \cdot g_1 \in \mathcal{O}(S). \quad (34)$$

In fact, if $\mathbf{n}_+ \geq 2$ holds, we have (the case $\mathbf{n}_- \leq -2$ follows analogously)

$$\begin{aligned} g_1 \cdot \iota(\Sigma_0) = \iota(\Sigma_1) & \xrightarrow{\text{Corollary 2.2}} g_1 \cdot \iota|_{\Sigma_{-1}} \sim_{\circ} \iota|_{\Sigma_2} \xrightarrow{(28)} \iota|_{\Sigma_{-1}} \sim_{\circ} g_1 \cdot \iota|_{\Sigma_2} \\ & \implies g_{-1} \cdot \iota|_{\Sigma_{-1}} \sim_{\circ} (g_{-1} \cdot g_1) \cdot \iota \xrightarrow{(28)} \iota \sim_{\circ} (g_{-1} \cdot g_1) \cdot \iota. \end{aligned}$$

In the last step, we have used that $g_{-1} \cdot \iota(\Sigma') = \iota(\Sigma_{-1})$, hence $\iota(\Sigma') = g_{-1} \cdot \iota(\Sigma_{-1})$ holds for some boundary segment Σ' of $\Sigma = \Sigma_0$.

Now, (32) is clear for $n = \pm 1$, as well as, by the right hand side of (29), for $n = -2$ and $n = 2$ if $\mathbf{n}_- \leq -2$ and $\mathbf{n}_+ \geq 2$ holds, respectively. Thus, if $\mathbf{n}_+ \geq 3$ holds (the case $\mathbf{n}_- \leq -3$ follows analogously), we can assume that (32) holds for all $1 \leq n \leq m$ for some $2 \leq m < \mathbf{n}_+$, and argue by induction. For this, let us first observe that then we have $h_{m-1} \cdot h_m \in \mathcal{O}(S)$, because

$$[h_{m-1} \cdot h_m] \stackrel{(12),(32)}{=} [g_1 \cdot q_{m-2}^+ \cdot h_2] \stackrel{(5)}{=} [g_1 \cdot h_2] \stackrel{(12)}{=} [g_1^2 \cdot (g_{-1} \cdot g_1)] \stackrel{(5)}{=} [g_{-1} \cdot g_1] \quad (35)$$

as $g_1^2 \in G_S$ holds by (33), and since $(g_{-1} \cdot g_1) \in \mathcal{O}(S)$ holds by (34). Thus, we have

$$[h_{m+1}] \stackrel{(29)}{=} [h_m \cdot (h_{m-1} \cdot h_m)] \stackrel{(12),(35)}{=} [h_m \cdot (g_{-1} \cdot g_1)] \stackrel{(12),(32)}{=} [g_1 \cdot (g_{-1} \cdot g_1)^m],$$

which proves the claim.

Step III

Finally, (26) holds for $n = \pm 1$, and follows inductively from (32) for each $n \in \mathfrak{n}$. In fact, we can assume that it holds for all $1 \leq n \leq m$ for some $1 \leq m < \mathfrak{n}_+$ (the other direction follows in the same way); and for $m = 2 \cdot k$ even, then we have

$$\begin{aligned} [g_{m+1}] &= [h_{m+1} \cdot g_m] \stackrel{(12), (32), (26)}{=} [g_1 \cdot (g_{-1} \cdot g_1)^{2k} \cdot (g_1 \cdot g_{-1})^k] \\ &= [g_1 \cdot (g_{-1} \cdot g_1)^k \cdot q_k^+] \stackrel{(33)}{=} [g_{\sigma(1)} \cdot \dots \cdot g_{\sigma(m+1)}]. \end{aligned}$$

Similarly, if $m = 2k + 1$ is odd, we obtain

$$\begin{aligned} [g_{m+1}] &= [h_{m+1} \cdot g_m] \stackrel{(12), (32), (26)}{=} [g_1 \cdot (g_{-1} \cdot g_1)^{2k+1} \cdot (g_1 \cdot g_{-1})^k \cdot g_1] \\ &= [(g_1 \cdot g_{-1})^{k+1} \cdot g_1 \cdot q_k^+ \cdot g_1] \stackrel{(33)}{=} [(g_1 \cdot g_{-1})^{k+1}] = [g_{\sigma(1)} \cdot \dots \cdot g_{\sigma(m+1)}]. \end{aligned}$$

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